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A New Approach to the Riemann Hypothesis

I. C. Chakravartty

1 Introduction

The Zeta-Function of Riemann is defined by the Dirichlet series:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

where $s = \sigma + it$, σ and t being real numbers.

The series is convergent and the function is analytic for $\sigma > 1$. It is a meromorphic function having a simple pole at $s = 1$ with residue 1. Although this function was known to Euler it was Riemann who discovered its most remarkable properties. Since then, this function has proved itself to be of fundamental importance, not only in the theory of prime numbers, but also in the higher theory of Gamma functions and allied functions.

It was first conjectured by Riemann [2] in 1859 that the complex (non-trivial) zeros of the Riemann zeta-function in the "critical strip" $0 < \sigma < 1$ lie on the "critical line" $\sigma = 1/2$. This conjecture has now become famous as the "Riemann Hypothesis". It is highly probable that Riemann's conjecture about the non-trivial zeros of the Riemann zeta-function is correct, but it has not been proved or disproved to this day. This famous hypothesis is now more than one hundred and thirty-five years old and many world-renowned mathematicians of successive generations devoted a considerable amount of time and energy to resolve this hypothesis, without any success so far. Some prominent mathematicians working on this subject now believe that there has to be some new approach to this problem. The object of this paper is to suggest a new approach to this age-old problem which, I hope, will someday lead to a solution to the problem.

2 Some Useful New Notations

It has long been known that there is a close connection between the prime numbers and the non-trivial zeros of the Riemann zeta-function. Riemann himself noted the existence of such a formula [5]. One such "explicit formula", given by Von-Mongoldt, is as follows [6]:

$$\begin{aligned}\psi(x) - x &= \sum_{n\leq x}{}' \Lambda(n) - x \\ &= \sum_{\rho} \frac{x^\rho}{\rho} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)}, \quad (x > 1) \qquad (1)\end{aligned}$$

In this formula, the notation Σ' is used to indicate that the term $n = x$ if it arises, is to be halved. Also, if n is a positive power of a prime, say $n = p^m$, p being a prime and m being a positive integer, then: $\Lambda_n = \log p$. Otherwise, $\Lambda_n = 0$. Finally, ρ runs through the complex zeros of $\zeta(s)$ and Σ_ρ is the principal value of:

$$\lim_{T\to\infty} \sum_{-T<\mathrm{Im}(\rho)<T}$$

Thus, the formula (1) connects a sum involving powers of primes and the imaginary parts of the zeros of the Riemann zeta-function.

In almost all standard texts and papers written on the subject of the Riemann hypothesis, the symbol ρ is used to denote the complex zeros of the Riemann zeta-function. This convention, however, has proved to be a disadvantage when it comes to getting a good insight into the problem of solving the Riemann hypothesis. Furthermore, I do believe that this convention actually hindered research in the subject. Instead of using ρ, we can, to our great advantage, use $1/2+i\gamma$ to denote a complex zero of the Riemann zeta-function, irrespective of whether the Riemann hypothesis is true or false. If the Riemann hypothesis is true then all the values of γ in $1/2+i\gamma$ are real numbers. On the other hand, if the Riemann hypothesis is false, then some of the values of γ in $1/2+i\gamma$ will be complex numbers. In either case, since complex zeros of $\zeta(s)$ occur in conjugate pairs, the series $-\sum_\rho \frac{x^\rho}{\rho}$ on the right-hand side of equation (1) becomes:

$$-\sum_\gamma \left(\frac{x^{\frac{1}{2}+i\gamma}}{\frac{1}{2}+i\gamma} + \frac{x^{\frac{1}{2}-i\gamma}}{\frac{1}{2}-i\gamma} \right) \tag{2}$$

To make this simpler from the viewpoint of calculations, we can replace $1/2 \pm i\gamma$ by $\pm i\gamma$ and x by e^γ. The expression in (2) then becomes:

$$\begin{aligned} & -\sum_\gamma \left(\frac{x^{i\gamma}}{i\gamma} + \frac{x^{-i\gamma}}{-i\gamma} \right) \\ = & \quad -\sum_\gamma \frac{e^{i\gamma y} - e^{-\gamma y}}{i\gamma} \\ = & \quad -\sum_\gamma \frac{\sin \gamma y}{\gamma}. \end{aligned}$$

We now consider a modification of (1) in the form:

$$\begin{aligned} & {\sum_{n\le x}}' \Lambda(n) n^{-k} - \frac{x^{1-k}}{1-k} \\ = & \quad -\sum_\rho \frac{x^{\rho-k}}{\rho-k} + \sum_{n=1}^{\infty} \frac{x-2n-k}{2n+k} - \frac{\zeta'(k)}{\zeta(k)}, \ (0<k<1, x>1). \end{aligned} \tag{3}$$

Putting $k=1/2$ and $x=e^y$, the first term on the right-hand side of (3) becomes:

$$\sum_{p^m<e^y} p^{-\frac{1}{2}m} \log p = {\sum_{m\log p\le y}}' p^{-\frac{1}{2}m} \log p.$$

The second term on the right-hand side of (3) becomes:

$$\begin{aligned}\sum_{n=.}^{\infty} \frac{x^{-(2n+\frac{1}{2})}}{2n+\frac{1}{2}} &= -2x^{-\frac{1}{2}} + \tan^{-1} x^{-\frac{1}{2}} - \frac{1}{2}\log\frac{1-x^{-\frac{1}{2}}}{1+x^{\frac{1}{2}}} \\ &= -2e^{\frac{1}{2}y} + \tan^{-1} e^{-\frac{1}{2}y} - \frac{1}{2}\log\tanh\frac{1}{4}y\end{aligned}$$

We now consider the functional equation of the Riemann zeta-function:

$$\zeta(s) = 2^s\pi^{s-1}\sin\frac{1}{2}s\pi\Gamma(1-s)\zeta(1-s).$$

Taking the logarithmic derivative of this equation and then putting $s = 1/2$, we obtain:

$$\frac{\zeta'\left(\frac{1}{2}\right)}{\zeta\left(\frac{1}{2}\right)} = -\frac{1}{2}C - \frac{1}{4}\pi - \frac{1}{2}\log 8\pi,$$

where C represents the Euler's constant. We thus obtain the following result from (3):

$$\begin{aligned}&\sum_{m\log p\leq y} p^{-\frac{1}{2}m}\log p - 2e^{\frac{1}{2}y} \\ &= -2\sum_{\gamma}\frac{\sin\gamma y}{\gamma} - \frac{1}{2}C - \frac{1}{4}\pi - \frac{1}{2}\log 8\pi - 2e \\ &\quad -\frac{1}{2}y + \tan^{-1} e^{-\frac{1}{2}y} - \frac{1}{2}\log\tanh\frac{1}{4}y, \quad (y>0).\end{aligned} \tag{4}$$

3 The Prime Number Summation Formula

The question of finding a "general", rather than an "explicit formula", connecting the non-trivial zeros of the Riemann zeta-function and the prime numbers have been the subject of research for a long time now. It was pointed out by Winter [8] in 1943 that this relationship appears to be of a reciprocal nature, analogous in some way to the "Fourier Inversion Formula". In 1948, A. P. Guinand came out for the first time with a general formula connecting the non-trivial zeros of the Riemann zeta-function and the prime numbers. This formula [3] actually connects the imaginary parts of the non-trivial zeros of the Riemann zeta-function and the logarithms of powers of primes as arguments of the functions concerned. With appropriate conditions on the functions $f(x)$ and $g(x)$, this general formula can be stated as follows:

$$\begin{aligned}\lim_{T\to\infty}&\left(\sum_{0<m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}}f(m\log p)-\int_0^T f(t)e^{\frac{1}{2}t}dt\right)\\&-\frac{1}{2}\int_0^\infty f(t)\left(\frac{1}{t}-\frac{e^{-\frac{3}{2}t}}{\sinh t}dt\right)\\&=-\sqrt{2}\pi\sum_\gamma g(\gamma)+\frac{1}{\sqrt{2}\pi}\int_0^\infty g(t)\log\frac{t}{2\pi}dt,\end{aligned}\tag{5}$$

where, p runs through the prime numbers, m through the positive integers and $\rho = 1/2+i\gamma$ through the non-trivial zeros of the Riemann zeta-function. It must be noted in this connection that if multiple zeros of $\zeta(s)$ exist, then the corresponding terms $g(\gamma)$ in this formula are to be multiplied by the order of the zeros.

Later, A. P. Guinand [4] discussed more general relationships of this type and showed that the relationship can be regarded as a type of Fourier reciprocity for almost periodic functions. In 1952, Weil [7] published a result for a wider class of zeta-functions.

However, all these results were proved assuming the Riemann hypothesis to be true. The question naturally arises, can we establish a general formula connecting the zeros of the Riemann zeta-function and the prime numbers without assuming the Riemann hypothesis? In 1970, I was able to prove a general summation formula [1], which, unlike Guinand's formula, was not based on the assumption that the Riemann hypothesis is true. This formula, therefore, is not dependent on any as yet unproven hypothesis. With appropriate conditions on the functions $\phi(x)$ and $\psi(x)$, this summation formula can be stated as follows:

$$\begin{aligned}
&\sum_{p,m} \frac{\log p}{p^{\frac{1}{2}m}} \phi(m \log p) - \int_0^\infty \phi(u) e^{\frac{1}{2}u} du \\
&- \frac{1}{2} \int_0^\infty \left(\frac{1}{u} - \frac{e^{-\frac{3}{2}u}}{\sinh u} du \right) \phi(u) du \\
= \quad & -\sqrt{2\pi} \left(\sum_{\mathrm{Re}\gamma > 0} \psi(\gamma) - \frac{1}{2\pi} \int_0^\infty \psi(u) \log \frac{u}{2\pi} du \right), \qquad (6)
\end{aligned}$$

where, $\phi(x) = \frac{\sqrt{2}}{\pi} \int_0^\infty \psi(t) \cos xt dt$, p runs through the prime numbers, m through the positive integers and $1/2 + i\gamma$ (where some of the values of γ are complex since we are not assuming the Riemann hypothesis) runs through the complex zeros of the Riemann zeta-function.

4 A New Perspective on the Riemann Hypothesis

We will now use a particular case of the Prime Number Summation Formula to gain a rather new perspective on the problem of solving the Riemann hypothesis.

Let

$$\begin{aligned}
f(x) &= \frac{1}{T} \left(e^{-\frac{1}{2}(x-z)^2/T^2} + e^{-\frac{1}{2}(x+z)^2/T^2} \right) \cos xy \\
g(x) &= e^{-\frac{1}{2}(x-y)^2T^2} \cos z(x-y) + e^{-\frac{1}{2}(x+y)^2T^2} \cos z(x+y)
\end{aligned}$$

With this pair, we find that the right-hand side of [5] can be split into:

$$-\sqrt{2}\pi \sum_\gamma e^{-\frac{1}{2}(\gamma-y)^2/T^2} \cos z(\gamma - y)$$

$$\text{and} \quad -\sqrt{2}\pi \sum_\gamma e^{-\frac{1}{2}(\gamma+y)^2/T^2} \cos z(\gamma + y)$$

We, therefore, obtain the following result from the summation formula:

$$\lim_{T\to\infty}\sum_{p,m} p^{-\frac{1}{2}m}\log p\left(e^{-\frac{1}{2}(z-m\log p)^2/T^2} + e^{-\frac{1}{2}(z+m\log p)^2/T^2}\right)\cos(y\,m\log p) = \left[\begin{array}{rl} -\sqrt{2}\pi, & \text{if } y=\pm\gamma \\ 0, & \text{elsewhere.}\end{array}\right. \tag{7}$$

Reversing the roles of $f(x)$ and $g(x)$, we obtain the following reciprocal formula:

$$\lim_{T\to\infty}\sqrt{2\pi}\sum_{\gamma}\left(e^{-\frac{1}{2}(\gamma-z)^2/T^2} + e^{-\frac{1}{2}(\gamma+z)^2/T^2}\right)\cos(y\gamma) = \left[\begin{array}{rl} p^{-\frac{1}{2}m}\log p, & \text{if } y=m\log p \\ 0, & \text{otherwise.}\end{array}\right. \tag{8}$$

From (7) and (8), we see that there exists a Fourier reciprocity between the weighted sequences $\{m\log p, p^{-\frac{1}{2}m}\log p\}$.

Formula (8) remains true even without the Riemann hypothesis. However, if the Riemann hypothesis is false, then the formula (7) needs modification if $1/2+iy$ is in the neighbourhood of an exceptional zero of $\zeta(s)$.

If the Riemann hypothesis is true, then the weighted sequence $\{m\log p, p^{-\frac{1}{2}m}\log p\}$ has just those fluctuations of spacings and weightings (and only those) permitted by a sequence of real frequencies $\{\gamma\}$. Consequently, any assumption that the Riemann hypothesis is false (i.e. some γ's are complex) should imply greater fluctuations in the former sequence.

The question then naturally arises: Are there limits to these fluctuations? There are. The values of $m\log p$ near $z=\log n$ can be spaced no less than the logarithms of consecutive integers near z. But if n is large, $\log(n+1)-\log n \simeq \frac{1}{n} = e^{-z}$. Therefore, the minimal spacings near z is about e^{-z}. It seems likely that the Riemann hypothesis depends on the limits of these fluctuations and that an investigation of these limits may some day lead to a solution of the Riemann hypothesis.

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Some Results of $M^{\times}/M/1$ Queue with Generalized Vacation

Gautam Choudhury

Abstract: The Laplace Stieltjes Transform (LST) for the joint distribution of the queue size and vacation period has been derived for an $M^{\times}/M/1$ queue with vacation.

Keywords: $M^{\times}/M/1$ Queue, Queue size and Vacation period.

1 Introduction

Recently, Borthakur and Choudhury (1997) have considered an $M^{\times}/G/1$ queue with generalized vacation, where the server is turned off each time as soon as the system becomes empty and he goes on vacation for a random length due to some uninterruptible task (e.g. machine breakdown, machine repair, scanning for new work etc.), which we shall refer to be a generalized vacation period. A generalized vacation starts at the instant the system becomes empty and the server leaves for the first vacation and then for a second vacation and so on until the server finds one or a batch of customers in the waiting line after returning from each vacation.

Due to the theoretical structures and applications in number of real life situations such as inventory/ production systems [e,g. see Lee et. al.

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(1994) and Choudhury (1997)], digital communication and data transmission systems [e.g. see Medhi (1997)], vacation queues become the subject of deep study in recent years. Hence, in this article we derive the LST of the joint distribution of the queue size and vacation period due to vacation for an $M^{\times}/M/1$ queue with generalized vacation.

2 Main Result

First of all let us define following notations:

λ - arrival rate,

μ - service rate,

X - arrival size random variable,

b_k - $P_r(X = k)$

$b_k^{(j)}$ - j-fold convolution of b_k with itself, and $b_k^{(0)} = 1$,

$X(z)$ - Probability generating function (PGF) of X,

V - vaction time random variable

$V(x)$ - Probability distribution function (PDF) of V,

$V^*(\theta)$ - LST of V and

$\rho = (\lambda E(X)/\mu)(< 1)$ is the utilization factor or offered of the system.

Borthakur and Choudhury (1997) have obtained the joint distribution of the queue size and vacation period $P_j(x)$ $(x > 0, j \geq 0)$ as a product of two independent random variables, which can be expressed as

$$P_j(x) = I_v(x)A_j(x)\ ;\ x > 0,\ j = 0, 1, 2, \ldots \tag{1}$$

where $I_v(x) = \frac{(1-\rho)[1-V(x)]}{E(V)}$ is the PDF of the length of a vacation period and

$$A_j(x) = P_r[j \text{ arrivals during } (0,x]] = \sum_{k=0}^{j} \frac{e^{-\lambda x}(\lambda x)^k}{k!} b_j^{(k)}.$$

Let $P^*(\theta;z)$ be the PGF of $\{P_j^*(\theta); j \geq 0\}$ [the LST of $P_j(x)$; $x > 0$, $j \geq 0$], then proceeding exactly in similar manner with Choudhury (1996) [see section 6], we get from equation (1)

$$\begin{aligned} P^*(\theta;z) &= \sum_{j=0}^{\infty} z^j P^*(\theta) \ \ (: |z| \leq 1 :) \\ &= \frac{(1-\rho)[1-V^*(\theta+\lambda-\lambda X(z))]}{E(V)[\theta+\lambda-\lambda X(z)]} \end{aligned} \tag{2}$$

Now if we take limit $\theta \to 0$ in equation (2), we get

$$\begin{aligned} \lim_{\theta \to 0} P^*(\theta;z) &= (1-\rho)[1-V^*(\lambda-\lambda X(z))]/E(V)[\lambda-\lambda X(z)] \\ &= P_V(z) \text{ say,} \end{aligned} \tag{3}$$

which is the PGF of the stationary queue size distribution due to server's vacation period. It may be noted here that this result is obtained by Borthakur and Choudhury (1997). In particular, if we take limit of $\rho \to 0$ (i.e. no offered load in the system) in equation (3), we get

$$\begin{aligned} \lim_{\rho \to 0} P_V(z) &= [1-V^*(\lambda-\lambda X(z))]/E(V)[\lambda-\lambda X(z)] \\ &= \zeta(z) \text{ say.} \end{aligned} \tag{4}$$

Hence the relationship between equation (3) and (4) is

$$P_V(z) = (1-\rho)\zeta(z).$$

Note that equation (4) represents the PGF of the number of customer that arrive during the residual vacation time for a multiple vacation of batch arrival queue and it is independent of rate of service [for instance see Lee et. al. (1994)].

Similarly, taking limit of $z \to 1$ in equation (2), we get

$$\lim_{z\to 1} P^*(\theta; z) = (1-\rho)[1-V^*(\theta)]/\theta E(V),$$

which is the LST of the PDF of the length of a vacation period.

It may be verified that the steady state probability that the server remains idle due to sever's vacation period is independent of the vacation time and equals $(1-\rho)$, which is the steady state probability of the server remains idle in a simple $M^{\times}/M/1$ (or more general $GI^{\times}/G/1$) queue. With simple algebra, we have

$$\begin{aligned} P_r[\text{The server remains idle}] &= \lim_{\substack{z\to 1\\ \theta\to 0}} P^*(\theta; z) \\ &= (1-\rho). \end{aligned}$$

Let L_v be the mean queue size during a vacation period when the system is idle, then differentiating equation (3) with respect to z and then taking limit $z \to 1$, we get on using L'Hospital's rule

$$\begin{aligned} L_v &= dP_V(z)/dz\Big]_{z=1} \\ &= \lambda(1-\rho)E(X)E(V_R), \end{aligned}$$

where $E(V_R) = E(V^2)/2E(V)$ is the mean of residual vacation time. Also, it is well known to us [e.g. see Medhi (1997), section-1.7] that

$$\begin{aligned} & P_r[\text{System is idle}]E(\text{residual vacation time}) \\ = & (1-\rho)E(V^2)/2E(V) \\ = & W_v \text{ (say)}, \end{aligned}$$

where W_v = Expected time in the queue due to server's vacation.

Thus we have

$$W_v = L_v/\lambda E(X)$$

This shows that

The mean waiting time due to server's vacation

$$= \frac{\text{(Mean queue size due to server's vacation)}}{\text{(arrival rate)}}$$

which justifies the Little's law.

Concluding remark

Since vacation time is independent of the rate of service or more specifically service time distribution, therefore equation (4) holds good for even more general $M^{\times}/G/1$ types of queueing system.

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Unsteady Rotational MHD Motion of a Dusty Viscous Fluid Contained in the Semi-infinite Circular Cylinder due to an Initially Applied Impulse on the Surface

U.N. Das and A. Aziz

Abstract: The unsteady rotational MHD motion of an electrically conducting dusty viscous fluid contained in the semi-infinite circular cylinder due to an initially applied impulse on the surface in the presence of a transverse magnetic field has been studied. Initially the liquid and dust particles are at rest. The motion is set up by an impulse of the shearing force on the surface. The governing equations subject to the appropriate boundary and initial conditions have been solved analytically. The effect of mass concentration of dust particles on the fluid flow in presence of magnetic field and the effect of the strength of magnetic field on velocity profiles at fixed time has been depicted graphically.

1 Introduction

Studies on the influence of dust particles on viscous fluid flows are of importance in petroleum industry and in the purification of crude oils. Other important applications involving dust particles in boundary layers include soil solvation by natural winds, lunar surface erosion by the exhaust of a landing vehicle and dust entrainment in a cloud formed during a nuclear explosion. The unsteady motion of fluid resulting due to the pure rotation of a solid boundary or due to the application of a uniformly distributed shear stress along a solid boundary is of both theoretical and practical significance in fluid mechanics.

A number of workers studied both steady and unsteady two dimensional axisymmetric rotational flow of a viscous fluid in view of its growing importance in viscous technical problems. Khamrui [1] analysed the slow steady motion of an infinite viscous flow due to rotation of a circular cylinder. Iben [2] considered the non stationary, plane circular-symmetric flow of a viscous fluid which forms itself within as well as outside a rotating infinitely long circular cylinder. Bhattacharyya [3] studied the rotational motion produced in an enclosed field, contained in a circular cylinder of infinite depth. The disturbance was generated on the surface of the fluid by an impulsive couple. Later Mukherjee and Bhattacharyya [4] studied the rotational flow of viscous fluid due to the rotation of a circular cylinder or by the action of shearing stress on the boundary. Saffman [5] studied the stability of the laminar flow of a dusty gas with uniform distribution of dust particles. Michael [6] considered the Kelvin-Helmholtz instability of the dusty gas. Michael and Miller [7] has discussed the motion of dusty gas enclosed in the semi-infinite space above a rigid plane boundary. Using the formulation of Saffman [5] many authors studied a number of gas problems and the results are well documented in a review by Marble [8]. Recently, Mandal, Mukherjee and Mukherjee [9] analysed the rotational motion of a dusty viscous fluid contained in the semi-infinite circular cylinder due to and initially applied impulse on the surface. In the present investigation we have extended the problem studied by Mandal et. al. [9] to the MHD case. The flow of the dusty gas is initiated by an impulsive shearing force n the surface in presence of a transverse magnetic field.

2 Formulation of the Problem

Following Saffman [5] the basic equation to represent the motion of a dusty fluid in presence of uniform magnetic field are given by

$$\rho\left[\frac{\partial\vec{u}}{\partial t}+(\vec{u}.\nabla)\vec{u}\right] = -\text{grad}\,p^{*}+\mu\nabla^{2}\vec{u}+KN(\vec{v}-\vec{u})+\vec{J}\times\vec{B} \tag{2.1}$$

$$m\left[\frac{\partial\vec{v}}{\partial t}+(\vec{v}.\nabla)\vec{v}\right] = K(\vec{v}-\vec{u}) \tag{2.2}$$

$$\text{div}\,\vec{u} = 0 \tag{2.3}$$

$$\frac{\partial N}{\partial t}+\text{div}\,N\vec{v} = 0 \tag{2.4}$$

where the velocities of the fluid and dust particles are $\vec{u}$ and $\vec{v}$ respectively. N is the number density of dust particles each of mass m. Here K is the Stokes coefficient of resistance (for spherical particles of radius d, it is $6\pi\mu d$), t, p^{*}, ρ, μ, the time, the pressure, the density and the viscosity of the fluid respectively. The last term on the right hand side of (2.1) represent the force on the fluid due to the interaction of magnetic induction $\vec{B}$ and the electric current J in the fluid.

In the present analysis, the following assumptions are made:

1. The dust particles are spherical in shape and are uniformly distributed.
2. The interaction between particles themselves is not considered.
3. The flow is fully developed.
4. The number density of dust particle is constant throughout the motion.
5. The applied magnetic field is considered to be along the axial direction of the cylinder.

6. The flow induced magnetic field is neglected.
7. There is no external applied electric field.
8. Only the electromagnetic body forces are present.
9. Fluid properties are invariable.

Initially the fluid and dust particles are at rest. Consider the flow of a dusty fluid in a long circular cylinder of radius a. Disturbance is set up by an impulse of the shearing force on the surface. Referring the problem to cylindrical polar co-ordinate (r, θ, z) we take the z-axis along the axis of the cylinder and the origin on the surface of the fluid. The prescribed impulsive shearing force will obviously generate a rotational velocity in the dusty fluid. Thus the fluid as well as dust particles will be moving in circular paths around the axis of the cylinder. Also a uniform magnetic field $\vec{B}(0, 0, B_0)$ is applied along the z-axis so that B_0 will be transverse to the flow field, r denotes the radial direction measured outward from the z-axis.

The symmetry consideration gives

$$\begin{aligned} &u_1 = u_3 = 0,\ v_1 = v_3 = 0,\ u_2 = u_2(r, z, t) \\ &v_2 = v_2(r, z, t) \text{ and } \frac{\partial}{\partial \theta} = 0 \end{aligned} \tag{2.5}$$

where u_2 and v_2 are circumferential velocities of liquid and dust particles respectively. Since the distribution of dust particles is uniform, the number density N of the particles equals N_0, a constant throughout the motion.

Then with the help of (2.5), equations (2.1) to (2.4) become

$$\begin{aligned} \rho \frac{\partial u_2}{\partial t} &= \mu \left[\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r}\frac{\partial u_2}{\partial r} - \frac{u_2}{r^2} + \frac{\partial^2 u_2}{\partial z^2} \right] \\ &\quad + K N_0 (v_2 - u_2) - \sigma B_0^2 u_2 \end{aligned} \tag{2.6}$$

$$m \frac{\partial v_2}{\partial t} = K(u_2 - v_2) \tag{2.7}$$

where σ is the electrical conductivity.

Initial boundary conditions are

$$u_2 = 0 \text{ at } t \leq 0 \text{ for all } z, v_2 = 0 \text{ at } t \leq 0 \text{ for all } z \tag{2.8}$$

$$P_{\theta z}]_{z=0} = \mu \left(\frac{\partial u_2}{\partial z}\right)_{z=0} \tag{2.9}$$

is prescribed as function of r and t for $r \leq a$, $t > 0$, where $P_{\theta z}$ is the shearing stress.

$$u_2 \longrightarrow 0 \text{ as } z \longrightarrow \infty \tag{2.10}$$

$$u_2]_{r=a} = 0, \;\; z \geq 0 \tag{2.11}$$

Introducing the following non-dimensional quantities $u = (u_2 a)/\nu$, $v = (v_2 a)/\nu$, $t_1 = t/\tau$, $r_1 = r/(\nu\tau)^{1/2}$, $z_1 = z/(\nu\tau)^{1/2}$, $a_1 = a/(\nu\tau)^{1/2}$, $f = (mN_0)/\rho$ (mass concentration of dust particles), $\tau = m/K$ (relaxation time of dust particles) in (2.6) to (2.11), we get (dropping suffices)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} + f.(v-u) - M^2 u \tag{2.12}$$

$$\frac{\partial v}{\partial t} = u - v \tag{2.13}$$

$$u = 0 \quad \text{at } t \leq 0 \text{ for all } z, v = 0 \text{ at } t \leq 0 \text{ for all } z \tag{2.14}$$

$$P_{\theta z}]_{z=0} = \left.\frac{\partial u}{\partial z}\right]_{z=0} = F(r).\delta(t) \text{ (prescribed)} \tag{2.15}$$

$$u \longrightarrow 0 \quad \text{as} \quad z \longrightarrow \infty \tag{2.16}$$

$$u]_{r=a} = 0, \;\; z \geq 0 \tag{2.17}$$

where $M = B_0 \left(\frac{\tau\sigma}{\rho}\right)^{1/2}$ is the Hartmann number and $\delta(t)$ is the Dirac's delta function.

3 Method of Solution

We solve the present problem by using technique of Laplace transform. Taking Laplace transform of equations (2.12), (2.13) and using (2.14), we get

$$
\begin{aligned}
p\overline{u} &= \frac{\partial^2\overline{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{u}}{\partial r}-\frac{\overline{u}}{r^2}+\frac{\partial^2\overline{u}}{\partial z^2}+f.(\overline{v}-\overline{u})-M^2\overline{u} && (3.1)\\
p\overline{v} &= \overline{u}-\overline{v} && (3.2)
\end{aligned}
$$

where $\overline{u}=\int_0^\infty ue^{-pt}dt, \overline{v}=\int_0^\infty ve^{-pt}dt$, Re$(p)>0$

Eliminating $\overline{v}$ from (3.1) and (3.2) we get

$$\frac{\partial^2\overline{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{u}}{\partial r}-\frac{\overline{u}}{r^2}+\frac{\partial^2\overline{u}}{\partial z^2}-\left[p\left(\frac{f}{p+1}+1\right)+M^2\right]\overline{u}=0 \qquad (3.3)$$

The solution of the above equation can be expressed in the form

$$\overline{u}(r,z,p)=\sum_{n=1}^{\infty}\overline{u}_n \qquad (3.4)$$

$$\text{where } \overline{u}_n=J_1(\alpha_n r)\overline{\phi}_n(z,p) \qquad (3.5)$$

and α_n can be determined form the equation (2.17) as the positive root of

$$J_1(\alpha_n a)=0 \qquad (3.6)$$

On substituting the value of $\overline{u}_n$ from (3.5) in (3.3) we get

$$\frac{d^2\overline{\phi}_n}{dz^2}-\left[\alpha_n^2+M^2+p\left(\frac{f}{p+1}+1\right)\right]\overline{\phi}_n=0 \qquad (3.7)$$

Solution of (3.7) subject to the Laplace transform of the condition (2.16) is

$$\overline{\phi}_n=A_n\exp\left[-z\left\{\alpha_n^2+M^2+p\left(1+\frac{f}{1+p}\right)\right\}^{1/2}\right] \qquad (3.8)$$

where A_n is independent of z for all n.

From (2.15) transformed shearing stress on the surface is given by

$$
\begin{aligned}
\overline{P}_{\theta z}]_{z=0} &= F(r),\ r\le a\\
&= \sum_{n=1}^{\infty}C_nJ_1(\alpha_n r) && (3.9)
\end{aligned}
$$

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where

$$C_n = 2a^{-2}[J_2(\alpha_n a)]^{-2}\int_0^a rF(r)J_1(\alpha_n r)dr \tag{3.10}$$

Again from (3.4), (3.5) and (3.8) we get

$$\overline{P}_{\theta z} = -\sum_{n=1}^{\infty} J_1(\alpha_n r)A_n\left[\alpha_n^2 + M^2 + p\left(1+\frac{f}{1+p}\right)\right]^{1/2} \tag{3.11}$$

we now compare the expressions (3.9) and (3.11) to obtain

$$A_n = -C_n\left[\alpha_n^2 + M^2 + p\left(1+\frac{f}{1+p}\right)\right]^{-1/2} \tag{3.12}$$

The expression for velocity becomes

$$\begin{aligned} u &= -\sum_{n=1}^{\infty} C_n J_1(\alpha_n r)L^{-1}\left[\left\{\alpha_n^2 + M^2 + p\left(1+\frac{f}{1+p}\right)\right\}^{-1/2}\right. \\ &\quad \left.\exp\left\{-z\left\{\alpha_n^2 + M^2 + p\left(1+\frac{f}{1+p}\right)\right\}^{1/2}\right\}\right] \end{aligned} \tag{3.13}$$

where L^{-1} is the inverse Laplace transformation operator.

Inverse Laplace transform of (3.13) presents some difficulties and we restrict ourselves to calculate the velocity expression for large values of time t only. For large time t, $p < 1$ and

$$\left[\alpha_n^2 + M^2 + p\left(1+\frac{f}{1+p}\right)\right]^{1/2} \simeq [\alpha_n^2 + M^2 + p(1+f)]^{1/2}$$

Then

$$\begin{aligned} & L^{-1}\left[\left\{\alpha_n^2 + M^2 + p\left(1+\frac{f}{1+p}\right)\right\}^{-1/2}\right. \\ & \left.\exp\left\{-z\left\{\alpha_n^2 + M^2 + p\left(1+\frac{f}{1+p}\right)\right\}^{1/2}\right\}\right] \\ = {} & \frac{1}{(1+f)^{1/2}}\exp\frac{-(\alpha_n^2+M^2)t}{1+f}\cdot\left\{(\pi t)^{-1/2}e^{-z^2(1+f)/(4t)}\right\} \end{aligned}$$

Equation (3.13) is then given by

$$u = -\frac{1}{[\pi t(1+f)]^{1/2}} \exp[-z^2(1+f)/(4t)] \times \sum_{n=1}^{\infty} C_n J_1(\alpha_n r) \exp\left[-\frac{(\alpha_n^2 + M^2)t}{1+f}\right] \tag{3.14}$$

This solution satisfies initial and boundary conditions given by (2.14) to (2.17)

4 Particular Cases

Case – 1: Motion due to impulsive shearing force applied within a circular area on the surface :

We take

$$\begin{aligned} F(r) &= \varepsilon r \quad \text{for} \quad 0 \le r \le b \\ &= 0 \quad \text{for} \quad b < r \le a \end{aligned} \tag{4.1}$$

Relation (4.1) corresponds to the situation where the applied force is acting within a circular area $r = b$, the rest of the surfaces being kept free from the impulse.

From (3.10) we get

$$C_n = 2\varepsilon b^2 J_2^{-2}(\alpha_n a) J_2(\alpha_n b)/(\alpha_n a^2)$$

on substituting the value of C_n in (3.14), we get the expression for velocity as

$$u = \frac{-2\varepsilon b^2}{a^2[\pi t(1+f)]^{1/2}} \cdot \exp[-z^2(1+f)/(4t)] \cdot \exp\left[\frac{-M^2 t}{1+f}\right] \times \sum_{n=1}^{\infty} J_2(\alpha_n b) J_2^{-2}(\alpha_n a) J_1(\alpha_n r) \alpha_n^{-1} \cdot \exp\left[\frac{-\alpha_n^2 t}{1+f}\right] \tag{4.2}$$

In absence of magnetic field ($M = 0$), the expression for velocity profile (equation (4.2)) becomes same as was deduced by Mandal [9] and in case when the fluid is clean ($f = 0$) and ($M = 0$) the expression (4.2) becomes same as was deduced by Bhattacharyya [3].

Case – *2*: Flow due to applied impulsive force distributed over the circumference of the circle $r = b$; $b < a$:

We take

$$F(r) = s\delta(r - b) \tag{4.3}$$

where s is constant and δ is Dirac delta function.

From (3.10) we get

$$C_n = 2a^{-2}sbJ_1(\alpha_n b)J_2^{-2}(\alpha_n a) \tag{4.4}$$

On substituting the value of C_n in (3.14) we get the expression for velocity as

$$\begin{aligned} u \;=\; & \frac{-2bs}{a^2[\pi t(1+f)]^{1/2}} \cdot \exp[-z^2(1+f)/(4t)] \cdot \exp\left[\frac{-M^2 t}{1+f}\right] \\ & \times \sum_{n=1}^{\infty} J_1(\alpha_n r)J_1(\alpha_n b)J_2^{-2}(\alpha_n a)\exp\left(\frac{-\alpha_n^2 t}{1+f}\right) \end{aligned} \tag{4.5}$$

In absence of magnetic field ($M = 0$), the expression for velocity (equation (4.5)) becomes same as that deduced by Mandal [9]. Again, when the fluid is clean and there is no magnetic field, that is, ($f = 0$) and ($M = 0$), the expression (4.5) becomes same as was deduced by Bhattacharyya [3].

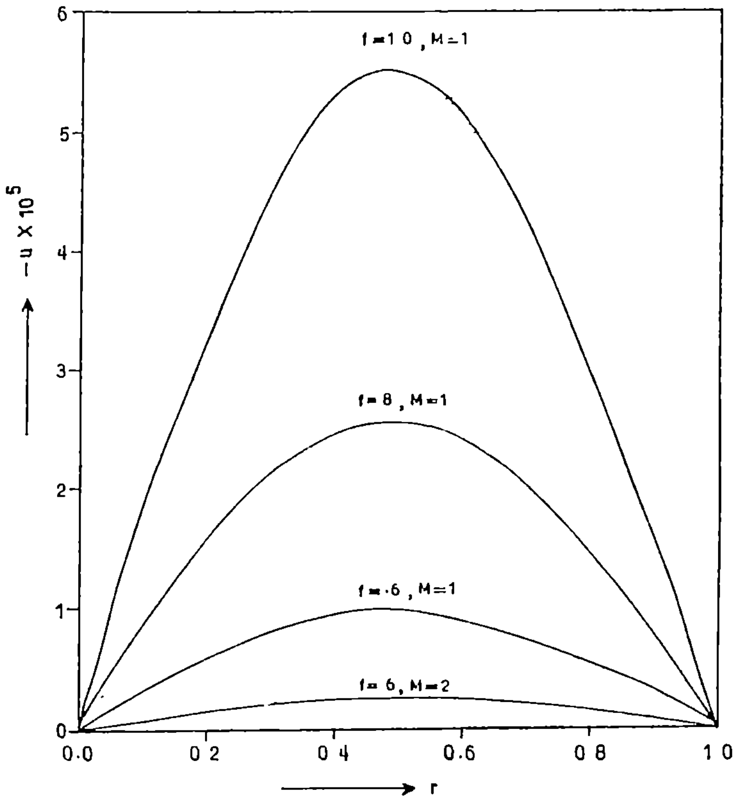

FIG. 1: VELOCITY DISTRIBUTION OF DUSTY FLUID FOR DIFFERENT VALUES OF f WHEN $t=1$, $\epsilon=1$, $z=1$, $a=1$, $b=0\ 5$, $M=1,2$.

5 Discussion of the Results

In order to illustrate the effects of mass concentration of dust particles on the flow field in presence of magnetic field and the effects of the transverse magnetic field on the velocity for two particular cases, numerical calculations are carried out for suitable values of the parameters entering the problem and depicted in figures (1) and (2). From figures 1 and 2 it is observed that magnitude of velocity of dusty fluid increases with the increase of mass concentration of dust particles for a fixed value of the Hartmann number M. From these figures it is also concluded that magnitude of velocity of dusty fluid decreases as the strength of the magnetic field increases for a fixed value of f.

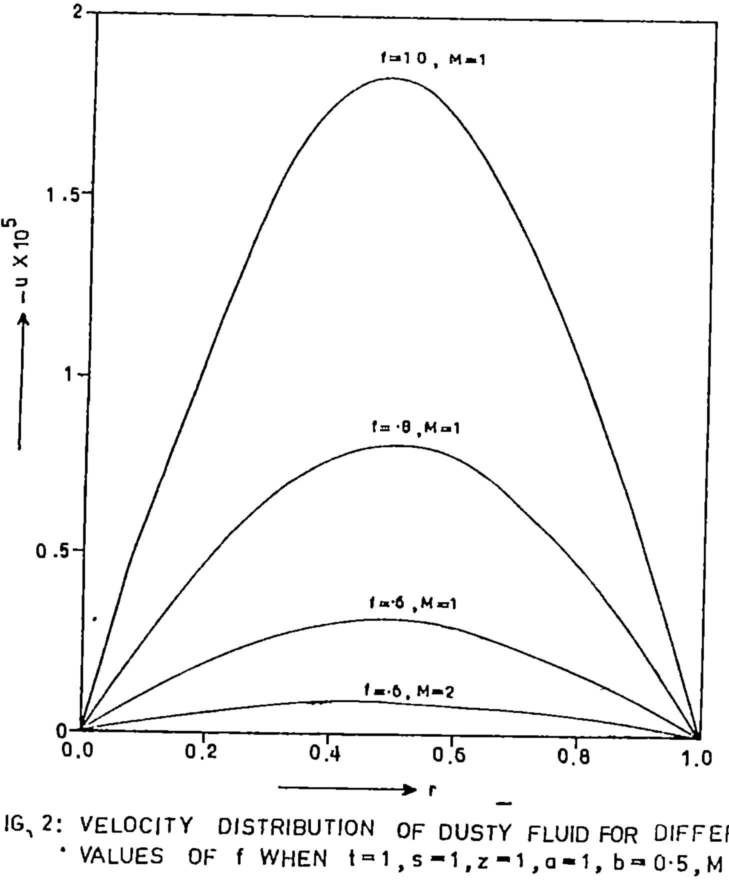

FIG. 2: VELOCITY DISTRIBUTION OF DUSTY FLUID FOR DIFFERENT VALUES OF f WHEN t=1, s=1, z=1, a=1, b=0·5, M=1, M=2.

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Weighted Composition Operators on $C_b(X, E)$

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On any space of functions with some structures there are two natural types of operators, one is the operator of multiplication and the other is the operator of composition. When we combine them somehow we get another type of operator known as the weighted composition operator. The operators belonging to this class has been the subject matter of systematic study for the last several decades on different function spaces, especially on L^p spaces, H^p spaces and spaces of continuous functions and have played a very significant role in the study of operators on Hilbert spaces. For a detailed information on such operators, we refer to the monograph [8] of singh and Manhas.

This paper is a continuation of the earlier paper [7] in which one of the authors has studied composition operators on $C_b(X, E)$. Here we shall report compact weighted composition operators on $C_b(X, E)$.

Preliminaries

Throughout this paper we shall assume, unless stated otherwise, that X is a completely regular Hausdorff space and E a non-trivial Banach space (over $\boldsymbol{K}$) with the norm $\|\,.\,\|$, where $\boldsymbol{K}$ is the field of reals ($\boldsymbol{R}$) or complex numbers ($\boldsymbol{C}$). Let $C(X, E)$ be the vector space of all continuous E valued

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functions on X, and let $C_b(X,E)$ be the subspace of $C(X,E)$ consisting of those functions which are bounded. Then $C_b(X,E)$ is a Banach space under the norm

$$\|f\|_\infty = \sup\{\|f(x)\| : x \in X\}.$$

In case $E = \boldsymbol{K}$, we simply write $C_b(X)$ instead of $C_b(X,\boldsymbol{K})$. For any f in $C_b(X)$ and any t in E, the function f_t defined by setting $f_t(x) = f(x)t$ for all x in X clearly belongs to $C_b(X,E)$. In particular, for $f = 1$, the constant one function on X, $1_t \in C_b(X,E)$. By $B_s(E)$ (respectively, $B_u(E)$) we denote the space $B(E)$ of all operators (i.e. bounded linear transformations) from E into itself when it is equipped with the strong (respectively, uniform) operator topology.

Weighted Composition Operators

Let π be a $B(E)$ valued function on X, T a selfmap on X and let $F(X,E)$ denote the space of all functions from X into E. Then the map $f \rightarrow \pi.f \circ T$ is a linear transformation from $C_b(X,E)$ to $F(X,E)$, where the composite product $\pi.f \circ T$ is defined as

$$\pi.f \circ T(x) = \pi(x)(f(T(x)))$$

for all $x \in X$. This linear transformation is denoted by $W_{\pi,T}$ and is called the weighted composition transformation induced by the pair (π, T). In case $W_{\pi,T}$ takes $C_b(X,E)$ into itself and is also bounded, it is called a weighted composition operator on $C_b(X,E)$.

If T is the identity map on X, the corresponding operator $W_{\pi,T}$ on $C_b(X,E)$ is denoted by M_π and is called the multiplication operator induced by π. On the other hand when π is the constant map and $\pi(x) = I$, the identity operator on E, for all $x \in X$, $W_{\pi,T}$ is denoted by C_T and is called the composition operator on $C_b(X,E)$ induced by T. For details, we refer to the recent survey article [9] of Singh and one of the authors.

Before characterizing those functions $\pi : X \to B(E)$ and $T : X \to X$ which induce weighted composition operators on $C_b(X,E)$, we first note the following:

If $W_{\pi,T}$ takes $C_b(X,E)$ into itself and $t \in E$, then $W_{\pi,T}1_t$ is an element of $C_b(X,E)$ and so we have

$$\begin{aligned} \sup\{\|\pi(x)t\| : x \in X\} &= \sup\{\|W_{\pi,T}1_t(x)\| : x \in X\} \\ &= \|W_{\pi,T}1_t\|_\infty < \infty \end{aligned}$$

Also, if $\pi(X)$ is a bounded subset of $B(E)$, we have

$$|||\pi||| = \sup\{\|\pi(x)\|_{B(E)} : x \in X\| < \infty.$$

Further, if $\{x_\alpha\}$ is a net in X such that $x_\alpha \to x$ in X, then we have

$$\|\pi(x_\alpha)t - \pi(x)t\| = \|W_{\pi,T}1_t(x_\alpha) - W_{\pi,T}1_t(x)\| \to 0$$

for all $t \in E$. This implies that the map $\pi : X \to B(E)$ is continuous in the strong operator topology, that is, $\pi \in C(X, B_s(E))$. Note that π is not necessarily continuous in the uniform operator topology (see Jamison and Rajagopalan [4] for example). The continuity of π in the strong operator topology shows that the set $N(\pi) = \{x \in X : \pi(x) \neq 0\}$ is open in X. Also, we shall see that the map T is continuous on $N(\pi)$. But T is not necessarily continuous on $X \setminus N(\pi)$ (see Jamison and Rajagopalan [4] again for example).

Theorem 1 *Let $W_{\pi,T}$ be a weighted composition operator on $C_b(X,E)$. Then*

(i) $\pi \in C(X, B_s(E))$,

(ii) *T is continuous on $N(\pi)$.*

Proof. Suppose that $W_{\pi,T}$ is an operator on $C_b(X,E)$, where $\pi : X \to B(E)$ and $T : X \to X$. Then, as noted above, $\pi \in C(X, B_s(E))$. Further to prove the condition (ii), we assume on the contrary that there exists an x_0 in $N(\pi)$ at which T is not continuous. Then there exists a net $\{x_\alpha : \alpha \in \Delta\}$ in $N(\pi)$ such that $x_\alpha \to x_0$ but $T(x_\alpha)$ does not converge to $T(x_0)$. This further implies that there exists an open neighbourhood G of $T(x_0)$ such that, for each $\alpha_0 \in \Delta$, $T(x_\alpha) \notin G$ for some $\alpha \geq \alpha_0$. Thus we can find a subnet $\{y_\beta\}$ of $\{x_\alpha\}$ such that $T(y_\beta) \notin G$. The complete regularity of X gives an $f \in C_b(X)$ such that $0 \leq f \leq 1$, $f(T(x_0)) = 1$ and $f(X \setminus G) = \{0\}$. Let t be a non-zero vector in E such that $\pi(x_0)t \neq 0$, and consider the function $f_t \in C_b(X,E)$ (as defined earlier). But for each β, we have

$$\begin{aligned}
&\|W_{\pi,T}f_t(y_\beta) - W_{\pi,T}f_t(x_0)\| \\
&= \|\pi(y_\beta)f_t(T(y_\beta)) - \pi(x_0)f(T(x_0))\| \\
&= \|\pi(x_0)t\| > 0.
\end{aligned}$$

This implies that $W_{\pi,T}f_t \notin C_b(X,E)$ which contradicts the fact that $W_{\pi,T}$ is an operator on $C_b(X,E)$. Therefore, T must be continuous on $N(\pi)$.

Remark. Conditions (i) and (ii) in Theorem 1 above are necessary but not sufficient for $W_{\pi,T}$ to be a weighted composition operator on $C_b(X,E)$ as the following example shows:

Example. Let $X = \mathbb{R}$, $E = L^2(\mathbb{R})$ (with Lebesgue measure),

$$T(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and $\pi(x) = \tau_x - I$, where $\tau_r - I$ is the operator of translation by x. Define

f in $C_b(X, E)$ as follows:

$$f(x) = \begin{cases} \sqrt{|x|}\chi_{(x, x+\frac{1}{|x|})} & \text{if} \quad |x| > 1. \\ \chi_{(x, x+1)} & \text{if} \quad 0 \leq |x| \leq 1 \end{cases}$$

Then note that $W_{\pi,T}f(0) = 0$ and $\|W_{\pi,T}f(\frac{1}{n})\|_2 = \sqrt{2}$, $n = 1, 2, \ldots$. This implies that $W_{\pi,T}$ is not an into map.

Remark. For sufficiency, we must assume that $\pi(X)$ is bounded and T be continuous on $\overline{N(\pi)}$.

Compact Weighted Composition Operators

Recently many authors have studied the compact weighted composition operators on continuous function spaces. Kamowitz [5] characterizes the compactness of these operators on the Banach algebra $C(X)$ of all complex valued continuos functions on a compact Hausdorff space X. His result has been extended to a larger class of spaces of continuous functions by Singh and Summers [10], and Jamison and Rajagopalan [4]. Further, Chan [1] and Takagi [11] have respectively improved upon the result of paper [4]. Feldman [2] deals with compact weighted composition operators on Banach lattices while Lindström and Llavona [6], and Gonzalez and Gutierrez [3] describe the relationship between compact and weakly compact homomorphisms (of composition type) between some algebras of continuous functions and differentiable functions, respectively.

Following Theorem 2 of Jamison and Rajagopalan [4], we shall present in the following theorem a characterization for compact weighted composition operators on $C_b(X, E)$. This theorem, in turn, extends Theorem 2 of Feldman [2] and a Theorem of Singh and Summers [10].

Theorem 2 *Let $W_{\pi,T}$ be a weighted composition operator on $C_b(X,E)$. Then $W_{\pi,T}$ is compact implies that the following conditions hold:*

(2.1) *π is continuous in the uniform operator topology;*

(2.2) *Each $\pi(x)$ is a compact operator on E; and*

(2.3) *$T(N(\pi,\epsilon))$ is finite for every $\epsilon > 0$*
where $N(\pi,\epsilon) = \{x \in X : \|\pi(x)\| \geq \epsilon\}$.

Proof. Suppose that $W_{\pi,T}$ is compact. Then the proof of condition (2.1) is the same as in the proof of Theorem 2 of [4]. To see that (2.2) holds, let $x_0 \in X$ be fixed and let $\{t_n\}$ be an arbitrary sequence in $E_0 = \{t \in E : \|t\| \leq 1\}$. For each n, consider the constant t_n-function 1_{t_n} (as defined earlier) in $C_b(X,E)$. Then $\{1_{t_n} : n \geq 1\}$ is a bounded sequence in $C_b(X,E)$, and the compactness of $W_{\pi,T}$ implies that $\{W_{\pi,T^{1_{t_n}}}\}$ has a convergent subsequence in $C_b(X,E)$. But $W_{\pi,T^{1_{t_n}}}(x_0) = \pi(x_0)t_n$. Therefore we conclude that $\pi(x_0)t_n$ also has a convergent subsequence in E. Thus $\pi(x_0)$ is a compact operator on E.

Next, assume that (2.3) is not satisfied. Then there exists an $\epsilon > 0$ and a sequence $\{a_i\}$ of distinct points in $T(N(\pi,\epsilon))$. As constructed in the proof of Theorem 1 of Feldman [2] we choose a collection of distinct neighbourhoods $\{N_j\}$ for points $\{a_{i_j}\}$ in $\{a_i\}$, and choose a collection of continuous functions $\{f_j\}$ such that $0 \leq f_j \leq 1$, $f_j(a_{i_j}) = 1$ and $f_j(X \setminus N_i) = \{0\}$. For non-zero vectors t_j to be chosen presently in E, define the function h_j as $h_j(x) = f_j(x)t_j$ for all $x \in X$ and $j \geq 1$. Then $\{h_j : j \geq 1\}$ is a bounded sequence in $C_b(X,E)$, and so by compactness of $W_{\pi,T}$, $\{W_{\pi,T^{h_j}}\}$ should have a convergent subsequence. But for $x \in X$ and t_j so chosen that $\|\pi(x)t_j\| \geq \epsilon$ and $T(x) = a_{i_j}$, we have

$$\begin{aligned} \|W_{\pi,T^{h_j}} - W_{\pi,T^{h_k}}\|_\infty &\geq \|\pi(x)h_j(T(x)) - \pi(x)h_k(T(x))\| \\ &= \|\pi(x)t_j\| \cdot |f_j(a_{i_j})| \geq \epsilon, \end{aligned}$$

a contradiction.

Remark. The condition of Theorem 2 above are necessary but not sufficient as can be seen by the example given below:

Example. Let $X = \{0, 1, 2, \ldots\}$ with the discrete topology and let E be a separable infinite dimensional Hilbert space with $e_0, e_1, e_2, \ldots$ as an orthonormal basis. Let $T : X \to X$ be given by $T(x) = 0$ for all $x \in X$; and for each x in X, define $\pi(x)v = \langle v, e_x \rangle e_x$ for all $v \in E$. Then the varification of conditions (2.1)–(2.3) are trivial. But $W_{\pi,T}$ is not compact because if we define $f_i : X \to E$ as $f_i(x) = e_i$ for all $x \in X$, where $i = 0, 1, 2, \ldots$; then $\{f_i\}$ is a bounded sequence in $C_b(X, E)$ with $\|f_i\|_\infty = 1$ and $\|W_{\pi,T} f_i - W_{\pi}, T f_j\|_\infty = 1$ for $i \neq j$.

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Generalized Monotone Operators and Complementary Problem

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Abstract: The concept of F-monotonicity was first introduced by Kato and this generalizes the notion of monotonicity introduced by Minty. The purpose of the present paper is to define various types of F-monotonicities, discuss relationship among them and study their applications to variational inequalities and complementary problem.

Keywords: Variational inequality, complementary problem, closed convex cone, monotone operators, coercive operators.

1 Introduction

Let X and Y be any two (real or complex) Banach spaces. Let F and T be two non-linear operators such that

$$F : D(F) = X \to R(F) \subset Y$$

and

$$T : D(T) \subset X \to R(T) \subset Y^*,$$

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where$D$ and R denote the domain and range of the operators, respectively. The concept of F-monotonicity of the operator T was introduced and discussed in Kato [2,3]. The usual concept of monotonicity introduced by Minty [5] and discussed by many others is a special case of this concept, the case for which F is the identity operator.

The purpose of this paper is to introduce various types of F-monotonicities and discuss the relationships among them. We first introduce the following definitions:

T is said to be F-monotone if

$$\operatorname{Re}(Tx - Ty, F(x-y)) \geq 0 \textit{ for all } x, y \in D(T)$$

and strictly F-monotone if

$$\operatorname{Re}(Tx - Ty, F(x-y)) > 0 \textit{ for all } x, y \in D(T), x \neq y$$

T is strongly F-monotone if there exists a constant $c > 0$ such that

$$\operatorname{Re}(Tx - Ty, F(x-y)) \geq c\|F(x-y)\|^2$$

for all $x, y \in D(T)$. T is called α-F monotone if there exists a continuous strictly increasing function $\alpha : [0,\infty) \to [0,\infty]$ with $\alpha(0) = 0$ and $\alpha(r) \to \infty$ as $r \to \infty$ such that

$$\operatorname{Re}(Tx - Ty, F(x-y)) \geq c\|F(x-y)\|\alpha(\|F(x-y)\|)$$

for all $x, y \in D(T)$.

T is F-coercive if

$$\operatorname{Re}\frac{(Tx, Ty)}{\|Fx\|} \to \infty \text{ as } \|Fx\| \to \infty.$$

Let $Y = X$ and F be the identity operator on X. Then the above concepts are just equivalent to monotonicity, strict monotonicity, strong monotonicity, α-monotonicity and coercivity in the usual sense.

Let X be reflexive, $Y = X^*$ and X^* be strictly convex. Let $F : X \to X^*$ be the duality map in the usual sense (which is well-defined if we assume $\|Fx\| = \|x\|$). In this case F-monotonicity of T (which has range space $Y^* = X$) means that T is accretive in the sense of Browder [1]. It may be noted here that in this case F is itself monotone in the usual sense.

Let K be a convex subset of X. Then

$$x \in K : (Tx, F(y - x)) \geq 0 \text{ for all } y \in K \tag{1}$$

will be called a generalized variational inequality and any $x \in K$ which satisfies (1) will be called a solution of (1). Similarly

$$x \in K : (Ty, F(x - y)) \geq 0 \text{ for all } y \in K \tag{2}$$

will be called another generalized variational inequality. Note that if F is the identity map in (1) and (2), then we get the usual variational inequalities. Let S_1 and S_2, respectively, denote the sets of solutions of (1) and (2).

Let K be a convex cone in X. Let K^* be a subset of Y^* defined by

$$K^* = \{y \in Y^* : (y, Fx) \geq 0 \text{ for all } y \in K\}.$$

Then

$$x \in K,\ Tx \in K^*,\ (Tx, Fx) = 0 \tag{3}$$

will be called a generalized complementary problem and any $x \in K$ which satisfies (3) will be called a solution of (3). Note that if F is the identity map, then (3) reduces to the usual complementary problem. Let C denote the set of all solutions of (3).

Before describing the results we first quote the following definitions which will be needed in the sequel. The mapping F is said to be positive homogeneous if

$$F(tx) = tF(x) \text{ for } t > 0$$

F is called symmetric if $F(x) = F(-x)$ and antisymmetric if $F(x) = -F(-x)$ for all $x \in D(F)$. F is said to be additive if

$$F(x+y) = F(x) + F(y) \text{ for } x, y \in D(F).$$

The mapping T is said to be hemicontinuous if $D(T)$ is convex and for any $x, y \in D(T)$, the map $t \to T(tx + (1-t)y)$ of $[0,1]$ to Y^* is continuous for the natural topology of $[0,1]$ and the weak topology of Y^*.

2 Results

Theorem 1 *If $T : D(T) \subset X \to Y^*$ is α-F monotone, then it is strictly F-monotone (hence F-monotone) and F-coercive. In particular, every strongly F-monotone operator is strictly F-monotone and F-coercive.*

Proof. It follows from the definition that every α-F monotone operator is strictly F-monotone. To prove F-coercivity let T be α-F-monotone. Then there exists a continuous strictly increasing function $\alpha : [0,\infty) \to [0,\infty]$ with $\alpha(0) = 0$ and $\alpha(r) \to \infty$ as $r \to \infty$ such that

$$\operatorname{Re}(Tx - T0, Fx) \geq \|Fx\|\alpha(\|Fx\|)$$

or

$$\operatorname{Re}(Tx, Fx) \geq \operatorname{Re}(T0, Fx) + \|Fx\|\alpha(\|Fx\|)$$

or

$$\frac{\operatorname{Re}(Tx, Fx)}{\|Fx\|} \geq \frac{\operatorname{Re}(T0, Fx)}{\|Fx\|} + \alpha(\|Fx\|)$$
$$\to \infty \text{ as } \|Fx\| \to \infty.$$

Therefore T is F-coercive and this completes the proof.

Theorem 2 *If F is antisymmetric and T is strictly F-monotone, then S_1 is either empty or a singleton.*

Proof. Assume, to the contrary, that x_1 and x_2 are in S_1. Then

$$\mathrm{re}(Tx_1, F(x_2 - x_1) \geq 0 \tag{4}$$

and

$$\mathrm{re}(Tx_2, F(x_1 - x_2) \geq 0 \tag{5}$$

since F is antisymmetric it follows from (4) that

$$\mathrm{re}(Tx_1, F(x_1 - x_2) \leq 0 \tag{6}$$

From (5) and (6) we obtain

$$\mathrm{re}(Tx_1 - Tx_2, F(x_1 - x_2) \leq 0 \tag{7}$$

Since T is strictly F-monotone, this is impossible unless $x_1 = x_2$, and this completes the proof of the theorem.

Theorem 3 *Let F be positive homogeneous, let T be F-monotone and hemicontinuous. Then*

$$S_1 = S_2.$$

Proof. Let $x \in S_1$. Since T is F-monotone

$$\mathrm{re}(Ty, F(y - x) \geq \mathrm{re}(Tx, F(y - x) \geq 0.$$

Hence $x \in S_2$.

Let $x \in S_2$. Let $y \in K$ be arbitrary. Since K is convex,

$$y_t = (1 - t)x + ty, \quad 0 < t < 1$$

is in K. Hence

$$(Ty_t, F(y_t - x) \geq 0.$$

But $y_t - x = t(y - x)$ and since F is positive homogeneous we have

$$(Ty_t, F(y - x) \geq 0$$

Since T is hemicontinuous we now have

$$(Tx, F(y - x) \geq 0$$

and hence $x \in S_1$. Thus $S_1 = S_2$ and this completes the proof.

Theorem 4 *Let K be a closed convex cone in X, F additive and antisymmetric. Then $C = S_1$.*

Proof. Let $x \in S_1$. Take $y = 0$. Then $(Tx, F(-x)) \geq 0$. Since F is antisymmetric we have $(Tx, F(x) \leq 0$. Take $y = 2x \in K$. Then $(Tx, Fx) \geq 0$. Thus $(Tx, Fx) = 0$. Assume, to the contrary, that $Tx \notin K^*$. Then there exists $y_0 \in K$ such that $(Tx, Fy_0) < 0$. Since $x \in S_1$ and F is additive we have

$$0 > (Tx, Fy_0) \geq (Tx, Fx) = 0.$$

This contradiction proves that $Tx \in K^*$ and hence $x \in C$. Thus $S_1 \subset C$. Since F is additive, it is obvious that $C \subset S_1$. Thus $C = S_1$ and this completes the proof.

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Hydromagnetic Divergent Channel Flow of a Newtonian Electrically Conducting Fluid

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Abstract: The two-dimensional boundary layer flow through a divergent channel of a Newtonian electrically conducting fluid in the presence of transverse magnetic field is studied. A special form of magnetic field is chosen so as to yield similarity equation. First, perturbation solutions for the case of large magnetic parameter are derived. Finally, a direct numerical solution of the resulting boundary value problem is obtained using a Runge-Kutta shooting algorithm with Newton iteration.

Keywords: Divergent Channel, Electrically Conducting fluid, Magnetic Field.

1 Introduction

The divergent flow problem between two non-parallel planes was solved by Jeffery [1] by reducing the problem to an elliptic integral equation. Srivastava [2] extended this problem to an electrically conducting fluid in the presence of transverse magnetic field. He found that with the application

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of magnetic field it is possible to have purely divergent flow without any secondary flow for greater angle between the two planes. Rosenhead [3] presented the solution of two-dimensional incompressible laminar flow in a diverging channel with impermeable wall. Terril [4] forwarded the solution obtained for small laminar flow through the divergent channel with suction at one wall and blowing at the other wall. Srivastava and Hazarika [5] discussed the problem [2] by shooting method.

A solution for the flow through a convergent channel which is based on the boundary layer (approximation) equation was first obtained by Pohlhausen [6]. The solution was performed in a cartesian Co-ordinate system hence there are two non-zero velocity components along two mutually parpendicular directions. Goldstein [7] presented an interesting discussion of this problem.

The present work deals with the flow of an electrically conducting fluid through a divergent channel which is based on boundary layer approximation in the presence of transverse magnetic field. Similarity solutions are possible only if the magnetic field takes a special form. Linearized solutions for the case of large magnetic parameters are presented. Finally, a direct numerical solution of the similarity equation is obtained using a Runge-Kutta Shooting algorithm with Newton iteration.

2 Basic Equation

The basic boundary layer equations for the steady two-dimensional flow of an electrically conducting, viscous, incompressible fluid in the presence of a magnetic field $B(x)$ are taken as (see, Djukic [8]); for Newtonian fluid)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x}+v\frac{\partial^2 u}{\partial y^2}+\frac{\sigma B^2(x)}{\rho}(U-u), \tag{2.2}$$

subject to the boundary conditions

$$y=0: u=0, v=0; y=\infty : u=U(x), \tag{2.3}$$

where u and v are the flow velocities in the $x-$ and $y-$ directions respectively: U is the velocity component outside the boundary layer, ρ the fluid density, ν the kinematic viscosity and σ the electrical conductivity. In equation (2.2), the secondary effects of magnetic induction are ignored, i.e. the induced magnetic field is neglected being small in comparison to the applied magnetic field. Furthermore, we assume that the external electric field is zero and the electric field due to polarization of charges is also negligible.

As in Sinha and Choudhury [9], the potential flow near the source is taken to be

$$U(x)=\frac{u_1}{x}, \tag{2.4}$$

which with $u_1 > 0$ represent two dimensional divergent channel flow and leads to similar solution.

We introduce the following change of variables (see, Schlichting [10])

$$\eta(x,y)=y\sqrt{\frac{U(x)}{\nu x}}=\frac{y}{x}\sqrt{\frac{u_1}{\nu}} \tag{2.5}$$

$$\psi(x,y)=\sqrt{U(x)\nu x F(\eta)}=\sqrt{\nu u_1}F(\eta) \tag{2.6}$$

We obtain the velocity components

$$u=\frac{\partial \psi}{\partial y}=U(x)F'(\eta),\ v=-\frac{\partial \psi}{\partial x}=\sqrt{\nu u_1}\frac{\eta}{x}F'(\eta) \tag{2.7}$$

It can be easily verified that the continuity equation (2.1) is identically satisfied. Similarity solution exists if the magnetic field has the special form (see, Chiam [11])

$$B(x) = \frac{B_0}{x} \tag{2.8}$$

Using (2.4)–(2.8) in (2.2) gives the following equation

$$F''' + F'^2 + M(1 - F') - 1 = 0 \tag{2.9}$$

subject to the boundary conditions

$$F(0) = 0,\ F'(0) = 0,\ F'(\infty) = 1 \tag{2.10}$$

where $M = \sigma B_0^2/\rho u_1$, is the hydromagnetic parameter.

3 Approximate Solution for Large M

We may obtain a linearized solution by considering the magnetic parameter M to be large. The procedure given below essentially follows Sapunkov's work [12] which was concerned with studying the problem of an electrically conducting non-Newtonian power-law fluid in the presence of transverse magnetic field which has recently been followed by Chiam [11] to present the Perturbation solution of hydromagnetic flow over a surface stretching with a power-law velocity distribution. We first assume

$$z = \sqrt{M}\eta,\ f(z) = \sqrt{M}F(\eta) \tag{3.1}$$

which implies

$$\frac{df}{dz} = \frac{dF}{d\eta} \tag{3.2}$$

Equation (2.9) then becomes

$$f'''(z) - f'(z) + 1 = \epsilon(1 - f'^2(z)) \tag{3.3}$$

where $\epsilon = 1/M$. The corresponding boundary conditions are

$$f(0) = 0,\ f'(0) = 0,\ f'(\infty) = 1. \tag{3.4}$$

The unknown function $f(z)$ is expanded in terms of powers of the small parameter ϵ as follows:

$$f(z) = f_0(z) + \epsilon f_1(z) + \epsilon^2 f_2(z) + \dots \tag{3.5}$$

Equation (3.3) with the help of (3.5), then gives the following set of equations:

$$f_0''' - f_0' = -1, \tag{3.6}$$

$$f_1''' - f_1' = 1 - f_0'^2, \tag{3.7}$$

$$f_2''' - f_2' = -2f_0'f_1'. \tag{3.8}$$

The above uncoupled linear equations can be readily solved in sequence. The appropriate boundary conditions are

$$f_0(0) = 0,\ f_0'(0) = 0,\ f_0'(\infty) = 1, \tag{3.9}$$

$$f_1(0) = 0,\ f_1'(0) = 0,\ f_1'(\infty) = 0, \tag{3.10}$$

$$f_2(0) = 0,\ f_2'(0) = 0,\ f_2'(\infty) = 0. \tag{3.11}$$

The solution of equations (3.6)–(3.8) satisfying the respective boundary conditions (3.9)–(3.11) are

$$f_0(z) = z + \exp(-z) - 1, \tag{3.12}$$

$$f_1(z) = \frac{1}{6}\{(4 + 6z)\exp(-z) + \exp(-2z) - 5\}, \tag{3.13}$$

$$\begin{aligned} f_2(z) = {} & \frac{1}{36}\{(18z^2 + 42z + 23)\exp(-z) \\ & + 12z\exp(-2z) + \exp(-3z) - 38\}. \end{aligned} \tag{3.14}$$

From these solutions, we can easily compute the approximate skin friction co-efficient which, in this case, is given by

$$f''(0) = 1 - \frac{2}{3}\epsilon - \frac{2}{9}\epsilon^2, \tag{3.15}$$

to second order in the small parameter ϵ.

4 Numerical Solution

A numerical solution of equation (2.9) with the boundary conditions (2.10) cannot be obtained because to integrate a third order ordinary differential equation by the standard numerical methods, three conditions must be prescribed at the initial point of integration. Since one condition is always missing at this point, i.e. the value of $F''(0)$ is not prescribed at the initial point of integration, so it is necessary to determine the value of the missing initial conditions by an appropriate numerical method. The shooting method [13,14] is a suitable method which enables to convert a boundary value problem to an initial value problem. Hence a direct numerical solution of equation (2.9) subject to boundary conditions (2.10) can be obtained by a shooting method using the Runge-Kutta algorithm with Newton iteration in double precision. The trouble of this procedure is that we have to make an initial guess for the value of $F''(0)$ to initiate the shooting process. The success of the procedure depends very much on how good the guess is. The process is frequently very sensitive to the starting value. This problem is made worse by the presence of the magnetic parameter M in the governing equation. In order to study the cases corresponding to many values of the parameter M, we would have to make a great number of guesses. Fortunately, we have computed the value of F' for various values of the magnetic parameter M.

5 Results and Discussion

The variations of dimensionless velocity F' with η across the boundary layer for different values of the magnetic parameter M, are shown in Fig. 1. It can be seen from Fig. 1 that F' decreases with the increase of the magnetic parameter M. As M increases, the Lorentz force which opposes the flow

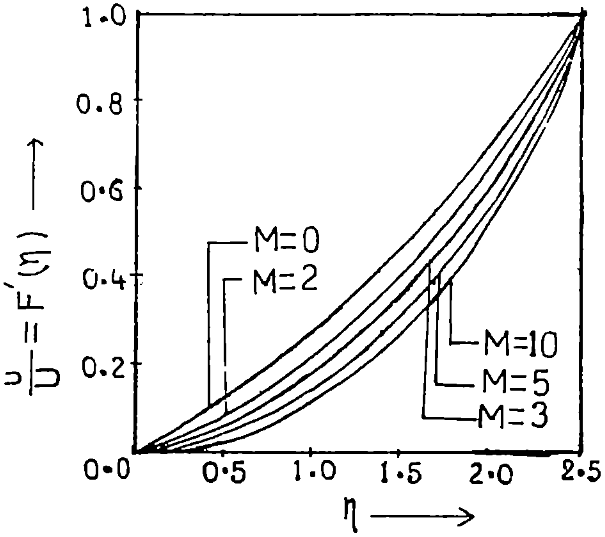

Fig. 1. The dimensionless velocity distribution F' for different values of magnetic parameter M.

also increases and leads to enhance deceleration of the flow. That is, the effects of the magnetic field are to decrease the velocity of the conducting fluid across the boundary layer.

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Fundamentals Concerning Water Waves: Part III : Solitary Waves

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[Introduction:

The contents of this paper is in continuation of the earlier two papers by the author published in Volume 2 (December 1995) and Volume 3 (December 1996) with the titles *Fundamentals Concerning Water Waves: Part I: Airy Waves* and *Fundamentals Concerning Water Waves: Part II: Stokes Waves* respectively. In the first paper the author investigated the surface gravity waves. There he considered the liquid (water) as inviscid and the motion irrotational so that a velocity potential exists. Under these assumptions the laws of motion in water for three dimensional motion were derived by taking into account both the horizontal and vertical accelerations. Several interesting results concerning surface gravity waves, progressive waves, surface waves and group velocity were established.

In the second paper, the author investigated various aspects of stokes waves. This is essentially a premise of the non-linear wave theory or finite amplitude waves as a departure from the small amplitude (linear theory) wave motion discussed in the first article. Here also the motion is considered as irrotational but the velocity potential Φ and the surface elevation η are expanded as series in terms of a suitable perturbation parameter ϵ (= ka which is related to the ratio of the wave height to the wavelength). Results have been obtained upto second order perturbation quantities. Finally, the stability aspects of the water waves have been shortly discussed. The present paper has been devoted to solitary waves pertaining to the shallow water waves. This requires dealing with non-linear theory of water waves. The author has presented a mathematical analysis for two dimensional case of shallow water and constant depth wave propagation.

– *U.N. Das*]

1 Nonlinear long waves in shallow water

In the Stokes second-order theory, the surface profile for small parameter kA (the wave steepness) has been obtained in (II:35). In very shallow water this Stokes wave profile becomes (see also Struik (1926)).

$$\eta(x,t) = A\sin(kx - \sigma t) - \frac{3A^2k}{4(kh)^3}\cos 2(kx - \sigma t) - \frac{kA^2}{4kh}. \tag{1}$$

when $kh \to 0$. We have seen in the previous section that for deep water waves when $kh \to \infty$ this surface profile is

$$\eta(x,t) = A\sin(kx - \sigma t) - \frac{A^2k}{2}\cos 2(kx - \sigma t). \tag{2}$$

From this analysis it is clear that the nonlinear theory of the Stokes expansion for $kA < 1$ is valid for deep water waves only. However, from this nonlinear theory in the case of shallow water waves, we have $\frac{3}{4}\frac{kA}{(kh)^3} < 1$ such that $(kA) < \left(\frac{4}{3}\right)(kh)^3$. This subsequently yields

$$\frac{A}{h} < \frac{4}{3}(kh)^2.$$

Thus for this case both $kh << 1$ and $\frac{A}{h} << 1$ must hold. Let us define these two parameters as

$$\gamma = kh << 1 \quad \text{and} \quad \delta = \frac{A}{h} << 1. \tag{3}$$

Here the second restriction is a much more severe one for many practical problems. Therefore, it is essential to investigate a nonlinear theory of shallow water waves. Historically, two different theories were developed, one by Airy (1845) and the second by Boussinesq (1877), which led to opposite conclusions. However, these differences were resolved by a fundamental paper by Ursell (1953). In particular, Ursell (1953) showed that the ratio,

$$U_r = \frac{\delta}{\gamma^2} = \frac{A/h}{(kh)^2} = \frac{kA}{(kh)^3} \tag{4}$$

which is known as the Ursell number, plays a very important role in deciding the choice of approximation.

2 Mathematical developments

A formal procedure for systematically developing shallow water wave theories was first introduced by Friedrichs (1948) and has also been outlined by Stoker (1957). The following mathematical equations for the constant depth case are mainly due to Benney (1966) and Peregrine (1967) and the analysis is made for the two-dimensional case. Because of the presence of two parameters, it will be convenient to express the equations in non-dimensional form prior to the perturbation procedure. For convenience we define the governing equations with their boundary conditions with upper case symbols for the dimensional case and lower case symbols for the non-dimensional case.

Laplace's equation:

$$\frac{\partial^2\Phi}{\partial X^2} + \frac{\partial^2\Phi}{\partial Z^2} = 0 \tag{5}$$

Dynamic surface boundary condition:

$$\frac{\partial\Phi}{\partial T} + g\tilde{\eta} + \frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial X}\right)^2 + \left(\frac{\partial\Phi}{\partial Z}\right)^2\right] = 0 \quad \text{at} \quad Z = \tilde{\eta} \tag{6}$$

Kinematic surface boundary condition:

$$\frac{\partial\tilde{\eta}}{\partial T} + \frac{\partial\Phi}{\partial X}\frac{\partial\tilde{\eta}}{\partial X} = \frac{\partial\Phi}{\partial Z} \quad \text{at} \quad Z = \tilde{\eta} \tag{7}$$

Bottom boundary condition

$$\frac{\partial \Phi}{\partial Z} = 0 \quad \text{at} \quad Z = -h \tag{8}$$

Here Φ = velocity potential, $\bar{\eta}$= wave elevation, X, Z = Cartesian coordinates, T = time, and g = acceleration due to gravity.

The linear theory suggests that the following dimensionless variables are appropriate to make the above equations dimensionless.

$$\left.\begin{aligned} &x = kX, \quad z = \tfrac{Z}{h}, \quad t = k\sqrt{gh}T \\ &\eta = \tfrac{\bar{\eta}}{A}, \quad \phi = \frac{\Phi}{\left(\frac{A}{kh}\right)\sqrt{gh}} \end{aligned}\right\} \tag{9}$$

The velocity components are normalized as

$$\left.\begin{aligned} &U = \tfrac{\partial \Phi}{\partial x} = \tfrac{A}{h}\sqrt{gh}\tfrac{\partial \phi}{\partial x} = \tfrac{A}{h}\sqrt{gh}u \\ &W = \tfrac{\partial \Phi}{\partial z} = \left(\tfrac{1}{kh}\right)\tfrac{A}{h}\sqrt{gh}\tfrac{\partial \phi}{\partial z} = \left(\tfrac{1}{kh}\right)\left(\tfrac{A}{h}\right)\sqrt{gh}w \end{aligned}\right\} \tag{10}$$

It is to be noted that the difference in scaling for horizontal and vertical components is required by continuity. The normalized equation and boundary conditions are:

Laplace's equation:

$$\gamma^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \qquad -1 < z < \delta\eta \tag{11}$$

Dynamic boundary condition:

$$\gamma^2 \left(\frac{\partial \phi}{\partial t} + \eta\right) + \frac{\delta^2}{2}\left[\gamma^2 \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] = 0, \qquad z = \delta\eta \tag{12}$$

Kinematic boundary condition:

$$\gamma^2 \left[\frac{\partial \eta}{\partial t} + \delta\left\{\frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}\right\}\right] = \frac{\partial \phi}{\partial z}, \qquad z = \delta\eta \tag{13}$$

Bottom boundary condition:

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = -1 \tag{14}$$

Keeping δ arbitrary, we first assume that $\gamma = kh << 1$. Since ϕ is analytic we can expand it as a power series in the vertical coordinate so that

$$\phi(x, z, t) = \sum_{n=0}^{\infty} (z+1)^n \phi_n(x, t) \tag{15}$$

Then we have

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= \sum_{n=0}^{\infty} (z+1)^n \frac{\partial \phi_n}{\partial x} \\
\frac{\partial^2 \phi}{\partial x^2} &= \sum_{n=0}^{\infty} (z+1)^n \frac{\partial^2 \phi_n}{\partial x^2} \\
\frac{\partial \phi}{\partial z} &= \sum_{n=0}^{\infty} n(z+1)^{n-1} \phi_n \\
&= \sum_{n=1}^{\infty} n(z+1)^{n-1} \phi_n \\
&= \sum_{n=0}^{\infty} (n+1)(z+1)^n \phi_{n+1} \\
\frac{\partial^2 \phi}{\partial z^2} &= \sum_{n=0}^{\infty} n(n+1)(z+1)^{n-1} \phi_{n+1} \\
&= \sum_{n=1}^{\infty} n(n+1)(z+1)^{n-1} \phi_{n+1} \\
&= \sum_{n=0}^{\infty} (n+1)(n+2)(z+1)^n \phi_{n+2}
\end{aligned}$$

Using these equations in Laplace's equation (11), we obtain

$$\gamma^2 \sum_{n=0}^{\infty} (z+1)^n \frac{\partial^2 \phi_n}{\partial x^2} + \sum_{n=0}^{\infty} (n+1)(n+2)(z+1)^n \phi_{n+2} = 0 \tag{16}$$

Since z is arbitrary, $-1 \leq z \leq \delta\eta$ the coefficient of each power of $(z+1)$ must vanish yielding a recurrence relation

$$\phi_{n+2} = -\frac{\gamma^2}{(n+1)(n+2)}\frac{\partial^2\phi_n}{\partial x^2}, \quad n = 0, 1, 2, 3, \ldots \tag{17}$$

But the bottom boundary condition implies that $\frac{\partial\phi}{\partial z} = 0$ at $z = -1$.

That is $\sum_{n=0}^{\infty}(n+1)(z+1)^n\phi_{n+1} = 0$ at $z = -1$. Corresponding to $n = 0, \phi_1 = 0$. It is clear from (17) that $\phi_1 = \phi_3 = \phi_{2n+1} = 0$. Also from (17) , we have

$$\left.\begin{array}{ll} n = 0: & \phi_2 = -\frac{\gamma^2}{1.2}\frac{\partial^2\phi_0}{\partial x^2} \\ n = 2: & \phi_4 = -\frac{\gamma^2}{3.4}\frac{\partial^2\phi_2}{\partial x} = \frac{\gamma^4}{4!}\frac{\partial^4\phi_0}{\partial x^4} \\ n = 4: & \phi_6 = -\frac{\gamma^2}{5.6}\frac{\partial^2\phi_4}{\partial x^2} = -\frac{\gamma^6}{6!}\frac{\partial^6\phi_0}{\partial x^6} \end{array}\right\} \tag{18}$$

If we assume that $\phi_0 = O(1)$ then it may be calculated that $\phi_2 = O(\gamma^2)$, $\phi_4 = O(\gamma^4)$, $\phi_6 = O(\gamma^6)$, and so on. Thus with an error of $O(\gamma^6)$, the velocity potential is

$$\phi = \phi_0 - \frac{\gamma^2}{2}(z+1)^2\frac{\partial^2\phi}{\partial x^2} + \frac{\gamma^4}{24}(z+1)^4\frac{\partial^4\phi_0}{\partial x^4} + O(\gamma^6) \tag{19}$$

Now we will investigate the dynamic and kinematic free-surface boundary conditions. Let us put $\zeta = 1 + \delta\eta$. With this substitution (12) reduces to

$$\begin{aligned} &\gamma^2\left[\frac{\partial\phi_0}{\partial t} - \frac{\gamma^2}{2}\zeta^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial\phi_0}{\partial t}\right) + \eta\right] \\ &+\frac{\delta}{2}\gamma^2\left[\left(\frac{\partial\phi_0}{\partial x}\right)^2 - \gamma^2\zeta^2\left(\frac{\partial\phi_0}{\partial x}\right)\left(\frac{\partial^3\phi_0}{\partial x^3}\right)\right] \\ &+\frac{1}{2}\delta\gamma^4\zeta^2\left(\frac{\partial^2\phi_0}{\partial x^2}\right)^2 \doteq O(\gamma^6) \end{aligned} \tag{20}$$

Equation (13) can be written as

$$\begin{aligned}
&\gamma^2\left[\frac{1}{\delta}\frac{\partial\zeta}{\partial t}+\frac{\partial\zeta}{\partial x}\left\{\frac{\partial\phi_0}{\partial x}-\frac{\gamma^2}{2}\zeta^2\frac{\partial^3\phi_0}{\partial x^3}\right\}\right]\\
&= -\gamma^2\zeta\frac{\partial^2\phi_0}{\partial x^2}+\frac{\gamma^4}{6}\zeta^3\frac{\partial^4\phi_0}{\partial x^4}+O(\gamma^6)
\end{aligned}\tag{21}$$

Now define $u_0 = \frac{\partial\phi_0}{\partial x}$ and then differentiating (20) with respect to x we obtain the dynamic boundary condition as

$$\begin{aligned}
&\left[\frac{\partial u_0}{\partial t}+\delta u_0\frac{\partial u_0}{\partial x}+\frac{1}{\delta}\frac{\partial\zeta}{\partial x}\right]+\gamma^2\frac{\partial}{\partial x}\left[-\frac{\zeta^2}{2}\frac{\partial}{\partial t}\left(\frac{\partial u_0}{\partial x}\right)\right.\\
&\left.-\frac{\delta}{2}\zeta^2 u_0\frac{\partial^2 u_0}{\partial x^2}+\frac{\delta}{2}\zeta^2\left(\frac{\partial u_0}{\partial x}\right)^2\right]=O(\gamma^4)
\end{aligned}\tag{22}$$

The kinematic boundary condition (13) can be simplified as

$$\begin{aligned}
&\frac{1}{\delta}\frac{\partial\zeta}{\partial t}+\frac{\partial\zeta}{\partial x}\left(u_0-\frac{\gamma^2}{2}\zeta^2\frac{\partial^2 u_0}{\partial x^2}\right)\\
&+\zeta\frac{\partial u_0}{\partial x}-\frac{\gamma^2}{6}\zeta^3\frac{\partial^2}{\partial x^2}\left(\frac{\partial u_0}{\partial x}\right)=O(\gamma^4)
\end{aligned}\tag{23}$$

Once η and u_0 are solved, the actual velocity components are given by

$$u = \frac{\partial\phi}{\partial x} = u_0-\frac{\gamma^2}{2}(z+1)^2\frac{\partial^2 u_0}{\partial x^2}+O(\gamma^4)\tag{24}$$

$$w = \frac{\partial\phi}{\partial z} = -\gamma^2(z+1)\frac{\partial u_0}{\partial x}+O(\gamma^4)\tag{25}$$

The pressure field is obtained from Bernoulli's equation whose dimensionless form is exactly

$$-p = z+\delta\left[\frac{\partial\phi}{\partial t}+\frac{\delta}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\frac{1}{\gamma^2}\left(\frac{\partial\phi}{\partial z}\right)^2\right\}\right]\tag{26}$$

where p has been normalized as $p=\frac{P}{\rho g h}$. The approximate pressure field may be obtained by using (24) and (25) into (26)

$$-p = z+\delta\left[\left\{\frac{\partial\phi_0}{\partial t}-\frac{\gamma^2}{2}(z+1)^2\frac{\partial^2 u_0}{\partial x\partial t}\right\}\right.$$

$$
\begin{aligned}
&+\frac{\delta}{2}\left\{u_0^2-\gamma^2(z+1)^2u_0\frac{\partial^2u_0}{\partial x^2}\right.\\
&\left.+\,\gamma^2(z+1)^2\left(\frac{\partial u_0}{\partial x}\right)^2\right\}\Bigg]+O(\gamma^4) \qquad (27)
\end{aligned}
$$

Equation (20) can be used to eliminate $\frac{\partial\phi_0}{\partial t}$ so that

$$
\begin{aligned}
p &= (\delta\eta-z)-\delta\frac{\gamma^2}{2}\{\zeta^2-(z+1)^2\}\left[\frac{\partial^2u_0}{\partial x\partial t}\right.\\
&\left.+\delta\left\{u_0\frac{\partial^2u_0}{\partial x^2}-\left(\frac{\partial u_0}{\partial x}\right)^2\right\}\right]+O(\gamma^4) \qquad (28)
\end{aligned}
$$

Now we can introduce the depth averaged horizontal velocity defined by

$$
\begin{aligned}
<u> &= \frac{1}{\zeta}\int_{-1}^{\delta\eta}\left(\frac{\partial\phi}{\partial x}\right)dz\\
&= \frac{1}{\zeta}\int_{-1}^{\delta\eta}\left(u_0-\frac{\gamma^2}{2}(z+1)^2\frac{\partial^2u_0}{\partial x^2}+\ldots\right)dz\\
&= u_0-\frac{\gamma^2}{6}\zeta^2\frac{\partial^2u_0}{\partial x^2}+O(\gamma^4) \qquad (29)
\end{aligned}
$$

which can be easily inverted to yield

$$
u_0=<u>+\frac{\gamma^2}{6}\zeta^2\frac{\partial^2}{\partial x^2}<u>+O(\gamma^4) \qquad (30)
$$

Substituting equation (30) into eqn (23) it follows that (kinematic boundary condition)

$$
\frac{\partial\zeta}{\partial t}+\delta\frac{\partial}{\partial x}(\zeta<u>)=O(\gamma^4) \qquad (31)
$$

which is just the depth averaged continuity equation.

Now expressing u_0 in terms of $<u>$, eqn (22) (dynamic boundary condition) and (28) (the pressure) can be written respectively, as

$$
\frac{\partial<u>}{\partial t} + \delta<u>\frac{\partial}{\partial x}<u>+\frac{1}{\delta}\frac{\partial\zeta}{\partial x}+\frac{\gamma^2}{6}\frac{\partial}{\partial t}\left(\zeta^2\frac{\partial^2<u>}{\partial x^2}\right)
$$

$$
\begin{aligned}
&+ \gamma^2 \frac{\partial}{\partial x}\left(-\frac{\delta}{3}\zeta^2 < u > \frac{\partial^2 < u >}{\partial x^2} + \frac{\delta}{2}\zeta^2\left(\frac{\partial < u >}{\partial x}\right)^2\right. \\
&- \left.\frac{\zeta^2}{2}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} < u >\right)\right) = O(\gamma^4) \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
p &= (\delta\eta - z) - \frac{\gamma^2}{2}\delta[\zeta^2 - (z+1)^2]\left[\frac{\partial^2 < u >}{\partial x \partial t}\right. \\
&\left. +\delta\left\{< u > \frac{\partial^2 < u >}{\partial x^2} - \left(\frac{\partial < u >}{\partial x}\right)^2\right\}\right] + O(\gamma^4) \qquad (33)
\end{aligned}
$$

These equations are valid for δ only. Extension to δ^2 is straightforward but tedious. Expressing equations (31), (32) and (33) in physical variables, we obtain

$$\frac{\partial}{\partial T}(h + \tilde{\eta}) + \frac{\partial}{\partial X}((h + \tilde{\eta}) < U >) = 0 \qquad (34)$$

$$
\begin{aligned}
\frac{\partial}{\partial T} \quad & < U > + < U > \frac{\partial}{\partial X} < U > + g\frac{\partial}{\partial X}(h + \tilde{\eta}) \\
& + \frac{1}{6}\frac{\partial}{\partial T}\left((h+\tilde{\eta})^2 \frac{\partial^2}{\partial X^2} < U >\right) \\
& + \frac{\partial}{\partial X}\left(-\frac{1}{3}(h+\tilde{\eta})^2(< U >)\left(\frac{\partial^2}{\partial X^2} < U >\right)\right. \\
& + \frac{1}{2}(h+\tilde{\eta})^2\left(\frac{\partial}{\partial X} < U >\right)^2 \\
& \left. - \frac{1}{2}(h+\tilde{\eta})^2\frac{\partial}{\partial X}\left(\frac{\partial}{\partial T} < U >\right)\right) = 0 \qquad (35)
\end{aligned}
$$

and

$$
\begin{aligned}
P &= \rho g(\tilde{\eta} - Z) - \frac{\rho}{2}\left[(h+\tilde{\eta})^2 - (Z+h)^2\right]\left[\frac{\partial}{\partial X}\frac{\partial}{\partial T} < U >\right. \\
& \left. + \left(< U > \frac{\partial^2}{\partial X^2} < U > - \left(\frac{\partial}{\partial x} < U >\right)^2\right)\right] \qquad (36)
\end{aligned}
$$

where

$$< U > = \frac{A}{h}\sqrt{gh} < u >, \frac{\partial}{\partial X} = k\frac{\partial}{\partial x} \qquad (37)$$

Limiting cases:

(A) Airy's theory for very long waves: $\gamma \to 0$ and $\delta = O(1)$

Airy's theory gives the leading-order approximation for very long waves of finite amplitude, because for very long waves $\gamma = kh \to 0$ and $\delta = \frac{A}{h} = O(1)$. Thus omitting terms proportional to γ^2 for equations (31), (32) and (33), we obtain in physical variables

$$\frac{\partial}{\partial T}(h+\tilde{\eta}) + \frac{\partial}{\partial X}\{(h+\tilde{\eta}) < U >\} = 0 \tag{38}$$

$$\frac{\partial}{\partial T} < U > + < U > \frac{\partial}{\partial X} < U > + g\frac{\partial}{\partial X}(h+\tilde{\eta}) = 0 \tag{39}$$

$$P = \rho g(\tilde{\eta} - Z) \tag{40}$$

These equations are valid for variable depth h(x) also. A distinguishing feature of Airy's approximation is that the pressure is hydrostatic.

(B)Boussinesq theory: $\delta = O(\gamma^2) < 1$

For weakly nonlinear and moderately long waves in shallow water $\delta = \frac{A}{h} < 1$ and $\gamma^2 = kh < 1$, eqns (31), (32) and (33) reduce to

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\{(1+\delta\eta) < u >\} = 0 \tag{41}$$

$$\frac{\partial < u >}{\partial t} + \delta < u > \frac{\partial}{\partial x} < u > + \frac{\partial \eta}{\partial x} - \frac{\gamma^2}{3}\frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial t} < u >\right) = 0 \tag{42}$$

$$p = (\delta\eta - z) + \frac{\gamma^2}{2}\delta(z^2 + 2z)\frac{\partial}{\partial x}\left\{\frac{\partial < u >}{\partial t}\right\} \tag{43}$$

In physical variables these can be written as

$$\frac{\partial \tilde{\eta}}{\partial T} + \frac{\partial}{\partial X}\{(\tilde{\eta} + h) < U >\} = 0 \tag{44}$$

$$\frac{\partial < U >}{\partial T} + < U > \frac{\partial}{\partial X} < U > + g\frac{\partial \tilde{\eta}}{\partial X}$$

$$-\frac{h^2}{3}\frac{\partial^2}{\partial X^2}\left\{\frac{\partial}{\partial T} < U >\right\} = 0 \tag{45}$$

$$P = \rho g(\tilde{\eta} - Z) + \frac{\rho}{2}(Z^2 + 2hZ)\frac{\partial}{\partial X}\left\{\frac{\partial < U >}{\partial T}\right\} \tag{46}$$

Equations (41), (42) and (43) or equivalently (44), (45) and (46) are called the Boussinesq equations. Here the pressure term is no longer hydrostatic. Moreover, the Airy and Boussinesq theories differ by the linear term multiplied by γ^2 in eqn (43). It is to be noted here that the two parameters $\delta = (\frac{A}{h})$ and $\gamma = kh$ can be treated as nonlinearity and dispersive parameters respectively. The Boussinesq theory accounts for the effects of nonlinearity δ and dispersion γ^2 to the leading order. However, when $\delta >> \gamma^2$, they reduce to Airy's equations which are valid for all δ; when $\delta << \gamma^2$ they reduce to the linearized approximation with weak dispersion. When $\delta \rightarrow 0$ and $\gamma^2 \rightarrow 0$, the classical linearized theory can be recovered. The foregoing analysis is valid for the constant depth case. For the variable depth case, this analysis can be extended easily (see Mei (1989)). Extensive work on the shallow water waves and the usefulness of the characteristic method has been reported by Rahman (1994) in his excellent book on water waves published by Oxford University Press and thus will not be repeated here.

3 Nonlinear dispersive waves

Mathematical expressions of the form $\exp\{i(kx - \sigma t)\} = \exp\{ik(x - ct)\}$ always represent wave propagating at a constant speed c called the phase speed. These waves are called progressive waves because they propagate without change of form. We know that the role of nonlinearity is to steepen a crest while dispersion tends to disperse into waves of different wavelengths. Thus the progressive waves have the role of dynamical equilibrium to balance these two effects.

Normalized Boussinesq equations (41) and (42) for one-dimensional waves can be rewritten by defining $< u >= \frac{\partial\phi}{\partial x}$ as

$$\frac{\partial\eta}{\partial t} + \delta\frac{\partial}{\partial x}\left(\eta\frac{\partial\phi}{\partial x}\right) + \frac{\partial^2\phi}{\partial x^2} = 0 \tag{47}$$

$$\frac{\partial\phi}{\partial t} + \eta + \frac{\delta}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 = \frac{\gamma^2}{3}\frac{\partial^3\phi}{\partial x^2\partial t} \tag{48}$$

Eliminating η from (47) and (48) we get a single equation in ϕ.

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = \frac{\gamma^2}{3}\frac{\partial^4\phi}{\partial^2 x\partial^2 t} - \delta\frac{\partial}{\partial t}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2\right] \tag{49}$$

Rewriting equation (49) in subscript notation for convenience, we obtain

$$\phi_{tt} - \phi_{xx} = \frac{\gamma^2}{3}\phi_{xxtt} - \delta\left(\phi_x^2 + \frac{1}{2}\phi_t^2\right)_t \tag{50}$$

It is this partial differential equation which we want to investigate for possible progressive wave solution. We now seek a solution of the following form:

$$\phi = f(\xi) \quad \text{with} \quad \xi = x - ct \tag{51}$$

Substituting this form of solution into equation (50), we obtain the following ordinary differential equation in f:

$$(c^2-1)f'' = \frac{\gamma^2}{3}c^2 f^{iv} + \delta c\left(1 + \frac{c^2}{2}\right)(f'^2)' \tag{52}$$

This equation implies that $c = 1 + 0(\delta, \gamma^2)$. Integrating once with respect to ξ yields

$$(c^2-1)f' = \frac{\gamma^2}{3}c^2 f''' + \delta c\left(1 + \frac{c^2}{2}\right)(f')^2 + B_1 \tag{53}$$

But from the first order theory (Airy's theory) we know that $\eta = -\phi_t = cf'$, and $c \neq O(1)$ where $c = \frac{C}{\sqrt{gh}}$. Hence without loss of generality, eqn

(53) can be rewritten as

$$(c^2-1)\eta = \frac{\gamma^2}{3}\eta'' + \frac{3}{2}\delta\eta^2 + B_1 \tag{54}$$

It is to be noted that

$$\eta = -\phi_t = cf', \eta' = c(cf')' = c^2 f'', \eta'' = c(c^2 f'')' = c^3 f'''.$$

Multiplying this equation by η' and integrating with respect to ξ again yields

$$(c^2-1)\frac{\eta^2}{2} = \frac{\gamma^2}{6}\eta'^2 + \frac{\delta}{2}\eta^3 + B_1\eta + B_2$$

which can be rewritten as

$$\frac{\gamma^2}{6}\eta'^2 = (c^2-1)\frac{\eta^2}{2} - \frac{\delta}{2}\eta^3 - B_1\eta - B_2 \tag{55}$$

where the integration constants B_1 and B_2 are both of $O(\delta, \gamma^2)$.

In the following we shall investigate two important analytical solutions of physical interest: (A) solitary waves, (B) cnoidal waves. Part III concerns with the solitary waves.

4 Solitary waves

A solitary wave has a single crest whose amplitude diminishes to zero as $|\xi| \to \infty$. Thus as $|\xi| \to \infty, \eta = \eta' = \eta'' = 0$ so that $B_1 = 0$ and $B_2 = 0$. Equation (55) simply becomes

$$(\eta')^2 = \frac{3\delta}{\gamma^2}\left(\frac{c^2-1}{\delta} - \eta\right)\eta^2 \tag{56}$$

Since the left-hand side of (56) is positive, the right-hand side must also be positive for a physical solution.

Therefore, we must have $c > 1$ or in physical variable, $C > \sqrt{gh}$, which is known as the super critical wave speed. Moreover, $\eta \leq \frac{c-1}{\delta}$ must also hold good. Hence $\frac{c-1}{\delta}$ is just the maximum amplitude of the crest which is unity because of normalization. Therefore

$$\frac{c^2-1}{\delta}=1 \quad \text{or} \quad c^2=1+\delta \tag{57}$$

In dimensional form (57) becomes

$$C=\sqrt{gh}\left(1+\frac{A}{h}\right)^{\frac{1}{2}} \quad \text{or} \quad C=[g(h+A)]^{\frac{1}{2}} \tag{58}$$

This relation was first found by Rayleigh (1877). Thus substituting (57) into (56) yields

$$\frac{d\eta}{d\xi}=\frac{(3\delta)^{\frac{1}{2}}}{\gamma}(1-\eta)^{\frac{1}{2}}\eta \tag{59}$$

which can be integrated to give

$$\frac{(3\delta)^{\frac{1}{2}}}{\gamma}(\xi-\xi_0)=-2\tanh^{-1}(1-\eta)^{\frac{1}{2}} \tag{60}$$

Thus after reduction it can be written as

$$\eta=\operatorname{sech}^2\left\{\frac{(3\delta)^{\frac{1}{2}}}{2\gamma}(\xi-\xi_0)\right\} \tag{61}$$

where ξ_0 is an integration constant and can be chosen to be zero. In dimensional form the surface profile can be written as

$$\tilde{\eta}=A\operatorname{sech}^2\left\{\frac{\sqrt{3}}{2}\left(\frac{A}{h^3}\right)^{\frac{1}{2}}(X-CT)\right\} \tag{62}$$

This is called the solitary wave of Boussinesq (1871, 1872). The solitary wave form is shown in Fig. 1. A therefore represents the height of the wave and h the depth at infinity.

Solitary waves can be easily generated in a wave tank by giving any kind of impulse. Munk (1949) has done extensive work for applications of solitary wave theory. Numerical solutions of solitary waves can be found

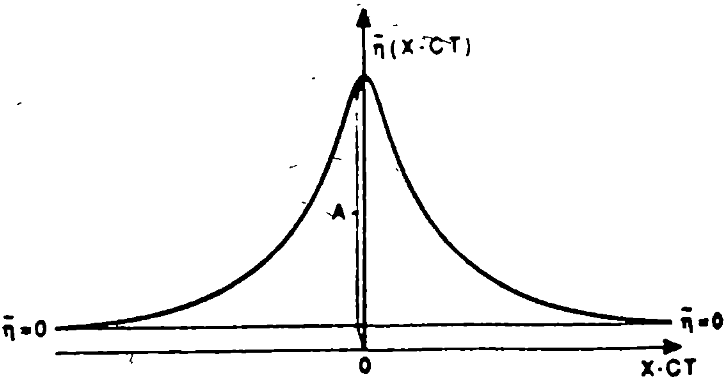

Figure 1: Dimensionless free surface profile of a solitary wave

in the work of Byatt-Smith (1970). Fenton (1972) extended the expansion procedure of Benjamin and Lighthill (1954) to the ninth order. In a series of remarkable papers, Longuet-Higgins (1973, 1974) and his associates made a very thorough investigation of solitary waves. Rahman (1994) recently presented a thorough investigation of the interaction of solitary waves. For further insight about this topic, interested readers are referred to the work of Whitham (1974).

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On the Pontryagin Maximum Principle for Constant – Time Linear Control Systems in Locally Convex Linear Topological Spaces

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Abstract: In this paper some dual characterization of linear systems in locally convex linear topological spaces, satisfying the Pontryagin maximum principle (briefly, PMP) are obtained. Several examples can be found (at least in normed linear spaces) in Rolewicz [7].

Keywords: Locally convex linear topological spaces, constant time linear control systems, Pontryagin maximum principle, smooth.

1 Introduction

Rolewicz [7] and Singer [9] have discussed constant time linear control system in Banach space setting. By a constant time linear control system in locally convex linear Housdorff topological vector spaces, we mean a system consisting of two real locally convex linear Housdorff topological vector spaces X, Y and a continuous linear mapping $F : X \to Y$. A linear system is said to satisfy the Pontryagin maximum principle (briefly PMP) if for each $y_0 (\neq \theta) \in F(X)$, there exists a continuous linear functional

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$g_0 \in Y^*$ such that

$$\begin{aligned}
&\inf_{x\in X}\left\{\sup_{x'\in X^*}|\langle x,x'\rangle| : F(x)=y_0, |\langle x,x'\rangle|\le 1\right\}\\
=\;&\inf_{x\in X}\left\{\sup_{x'\in X^*}|\langle x,x'\rangle| : g_0(F(x))=g_0(y_0), |\langle x,x'\rangle|\le 1\right\} \qquad (1)
\end{aligned}$$

where X^*, Y^* be the adjoint (conjugate) space to X and Y respectively and θ denotes the null element.

Our problem is to determine some sufficient condition for a linear system to satisfy *PMP* in locally convex linear topological spaces over the field of real numbers. For this we require the following theorems, and definitions.

Definition 1 *We define $G^\perp$ by*

$$G^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in G\}, \qquad (2)$$

where G is a linear subspace of a locally convex linear topological space X.

Theorem 1 *The linear system $(X \xrightarrow{F} Y)$ satisfies PMP iff corresponding to each $x_0(\neq\theta) \in X$, there exists $g_0 \in Y^*$ such that*

$$\begin{aligned}
&\sup_{f\in(kerF)^\perp}\left\{|f(x_0)| : \sup_{x\in X\subset X^{**}}|\langle f,x\rangle| = 1\right\}\\
=\;&\frac{|\langle F^*(g_0),x_0\rangle|}{\sup\limits_{x\in X\subset X^{**}}|\langle F^*(g_0),x\rangle|} \qquad (3)
\end{aligned}$$

F^* denotes the adjoint operator to F.

Proof : Let $y_0 = F(x_0)$. Then, by Hahn-Banach theorem, (Page 21, [5]), we get

$$\inf_{x\in X}\left\{\sup_{x'\in X^*}|\langle x,x'\rangle| : F(x)=y_0\right\}$$

$$\sup_{x'\in(X/KerF)^*}|\langle x_0+KerF,x'\rangle|$$

$$= \sup_{f \in (kerF)^{\perp}} \left\{ |\langle x_0, f \rangle| : \sup_{x \in X \subset X^{**}} |\langle f, x \rangle| = 1 \right\}. \tag{4}$$

Again, for any $g_0 (\neq \theta) \in Y^*/KerF^*$, the set

$$H = \{x \in X : |\langle F^*(g_0), x \rangle| = g_0(y_0)\} \tag{5}$$

is a hyperplane. Similarly, for any such g_0, we get,

$$\begin{aligned} &\inf_{x \in X} \left\{ \sup_{x' \in X^*} |\langle x, x' \rangle| : g_0(F(x)) = g_0(y_0) \right\} \\ &= \sup_{x' \in (X/KerF^*(g_0))^*} |\langle x_0 + KerF^*(g_0), x' \rangle| \\ &= \frac{|\langle F^*(g_0), x_0 \rangle|}{\sup\limits_{x \in X \subset X^{**}} |\langle F^*(g_0), x \rangle|}, \end{aligned} \tag{6}$$

Now if $(X \xrightarrow{F} Y)$ satisfies the PMP and $x_0 \in X$, $y_0 = F(x_0) \neq 0$, then, by (1), $g_0(y_0) \neq 0$, so $F^*(g_0) \neq 0$; hence by (1), (4), (6) we obtain (3).

Conversely if the condition (i) is satisfied and $y_0 \in F(X)$, then, corresponding to any $x_0 \in X$ such that $y_0 = F(x_0)$, from (3), (4), (6) we obtain (1). Hence $(X \xrightarrow{F} Y)$ satisfies the PMP, which completes the proof.

Several equivalent conditions may be obtained in the following way:

Corollary 1 *For a linear system $(X \xrightarrow{F} Y)$, the following conditions are equivalent:*

(*i*) $(X \xrightarrow{F} Y)$ *satisfies the PMP;*

(*ii*) *for each $x_0 (\neq \theta) \in X$ with $\sup_{x' \in X^*} |\langle x_0, x' \rangle| = 1$, there exists $g_0 \in Y^*$ with $\sup_{x \in X \subset X^{**}} |\langle F^*(g_0), x \rangle| = 1$ such that*

$$\begin{aligned} &\sup_{f \in (kerF)^{\perp}} \left\{ |f(x_0)| : \sup_{x \in X \subset X^{**}} |\langle f, x \rangle| = 1 \right\} \\ &= |\langle F^*(g_0), x_0 \rangle| \end{aligned} \tag{7}$$

(*iii*) *For each* $x_0 (\neq \theta) \in X$ *with*

$$\sup_{x' \in (X/KerF)^*} |\langle x_0 + KerF, x' \rangle| = 1$$

there exists $g_0 \in Y^*$ *with*

$$\sup_{x \in X \subset X^{**}} |\langle F^*(g_0), x \rangle| = 1,$$

such that

$$|\langle F^*(g_0), x_0 \rangle| = 1 \tag{8}$$

Corollary 2 *If a linear system* $(X \xrightarrow{F} Y)$ *satisfies the PMP, it can be shown that the image* V *of* $F^*(Y^*)$ *in* $(X/KerF)^*$*, under the topological isomorphism* $(KerF)^\perp \equiv (X/KerF)^*$*, is a subspace of characteristic I, i.e.,*

$$\begin{aligned} &\sup_{x' \in (X/KerF)^*} |\langle x_0 + KerF, x' \rangle| \\ = &\sup_{\psi \in V} \left\{ |\langle x_0 + KerF, \psi \rangle| : \sup_{x \in X \subset X^{**}} |\langle \psi, x \rangle| = 1 \right\} \end{aligned} \tag{9}$$

$x_0 + KerF \in X/KerF$, [4] (assuming $KerF$ is a closed subspace of Y).

Proof: If the PMP holds, then by $F^*(Y^*) \subset (KerF)^\perp$ and by Theorem 1, for each $x_0 \in X$ we have

$$\begin{aligned} &\sup_{x' \in (X/KerF)^*} |\langle x_0 + KerF, x' \rangle| \\ = &\sup_{f \in (kerF)^\perp} \left\{ |\langle x_0, f \rangle| : \sup_{x \in X \subset X^{**}} |\langle f, x \rangle| = 1 \right\} \\ = &\sup_{f \in F^*(Y^*)} \left\{ |\langle x_0, f \rangle| : \sup_{x \in X \subset X^{**}} |\langle f, x \rangle| = 1 \right\} \end{aligned} \tag{10}$$

Hence, the topological isomorphism $f \to \psi$ of $(KerF)^\perp$ onto $(X/KerF)^*$ is given by

$$|\langle x + KerF, \psi \rangle| = |\langle x, f \rangle|, \; x + KerF \in X/KerF \tag{11}$$

we obtain (9), which completes the proof.

Theorem 2 *The linear system* $(X \xrightarrow{F} Y)$ *satisfies the PMP, if* $F(X)$ *is closed in* Y.

Proof: If $x_0(\neq \theta) \in X$, then, there exists $f_0 \in (KerF)^{\perp}$ such that

$$\sup_{x' \in (X/KerF)^*} |\langle x_0 + KerF, x' \rangle| = \frac{|\langle f_0, x_0 \rangle|}{\sup\limits_{x \in X \subset X^{**}} |\langle f_0, x \rangle|'} \tag{12}$$

Since $F(X)$ is closed in Y, then $(KerF)^{\perp} = F^*(Y^*)$, by [4]. Hence $f_0 = F^*(g_0)$ for some $g_0 \in Y^*$. Using (12) we get (3), by Theorem 1, the linear system $(X \xrightarrow{F} Y)$ satisfies the PMP.

Definition of Polar Set: Let X be a locally convex linear topological space and M be a bounded subset of X. Then we shall define the polar set M^{π} of the set $M \subseteq X$ in X^* by $\{f \in X^* : \sup_{x \in M}\{|\langle x, f \rangle|\} \leq 1\}$, (Page 136, [10]).

Again let E_{π} be the polar set of $E \subseteq X^*$ in $X \subset X^{**}$ [10]. Then the boundary of E_{π} will be denoted by ∂E_{π} and is defined as follows:

$$\partial E_{\pi} = \left\{ x \in X : \sup_{f \in E}\{|\langle x, f \rangle| = 1\} \right\}$$

Definition of smooth: A locally convex linear topological space X is smooth, if at each point $x \in \partial E_{\pi}$ there is one and only one supporting hyperplane of E_{π}, [2].

Theorem 3 *Let* $(X \xrightarrow{F} Y)$ *be a linear system, with* F *one to one. Then,*

(i) if $M^{\pi} \subset F^*(Y^*)$, *then* $(X \xrightarrow{F} Y)$ *satisfies the PMP.*

(ii) if X *is smooth, the converse also holds.*

Proof: (i) If $x_0(\neq \varphi) \in X$, $\sup_{f \in X^*} |\langle x_0, f \rangle| = 1$, then, there exists $f_0 \in X^*$ with $\sup_{x \in X \subset X^{**}} |\langle f_0, x \rangle| = 1$, such that $|\langle f_0, x_0 \rangle| = 1$, [5]. Hence

$f_0 \in M^\pi$. But, if $M^\pi \subset F^*(Y^*)$, then $f_0 = F^*(g_0)$ for some $g_0 \in Y^*$. Thus, $\sup_{x\in X\subset X^{**}} |\langle F^*(g_0), x\rangle| = 1$, $|\langle F^*(g_0), x\rangle| = \sup_{f\in X^*} |\langle x_0, f\rangle| = \sup_{f\in(X/KerF)^*} |\langle x_0 + KerF, f\rangle|$ (since F is one-to-one). Hence, by Corollary 1, $(X \xrightarrow{F} Y)$ satisfies the PMP.

(ii) Let X be smooth and that $(X \xrightarrow{F} Y)$ satisfies the PMP, and let $f \in M^\pi$ with $\sup_{x\in X\subset X^{**}} |\langle f, x\rangle| = 1$. Then by the definition of M^π, there exists $x_0 \in X$ with $\sup_{f\in X^*} |\langle x_0, f\rangle| = 1$, such that $|\langle x_0, f\rangle| = 1$. Furthermore, by the PMP and by Corollary 1, for this x_0 there exists $g_0 \in Y^*$ with $\sup_{x\in X\subset X^{**}} |\langle F^*(g_0), x\rangle| = 1$, such that

$$\sup_{f\in X^*} |\langle x_0, f\rangle| = \sup_{f\in(X/KerF)^*} |\langle x_0 + KerF, f\rangle| = |\langle F^*(g_0), x_0\rangle| = 1.$$

(since F is one-to-one). Since X is smooth, we must have $f = F^*(g_0) \in F^*(Y^*)$. Hence, since $f \in M^\pi$ is arbitrary with $\sup_{x\in X\subset X^{**}} |\langle f, x\rangle| = 1$, we obtain by homogeneity, that $M^\pi \subset F^*(Y^*)$, which completes the proof.

Corollary 3 *Let $(X \xrightarrow{F} Y)$ be a linear system, with X smooth and F one-to-one. Then,*

- *(i) if $(X \xrightarrow{F} Y)$ satisfies the PMP, $\overline{F^*(Y^*)} = X^*$ (in the strong topology);*
- *(ii) if $(X \xrightarrow{F} Y)$ satisfies the PMP and X is semi-reflexive, $F^*(Y^*) = X^*$, and hence $F(X)$ is closed in Y;*
- *(iii) If X^* is non-separable and F is compact $(X \xrightarrow{F} Y)$ does not satisfy the PMP.*

Proof: (i) Clearly $\overline{M^\pi} = X^*$, by the Bishop – Phelps theorem [1]. Since $(X \xrightarrow{F} Y)$ satisfies the PMP, (by hypothesis), then, by Theorem 3 (ii), $M^\pi \subset F^*(Y^*)$. Consequently, $X^* = \overline{M^\pi} \subset \overline{F^*(Y^*)}$, so $\overline{F^*(Y^*)} = X^*$.

(ii) Since X is semi-reflexive, so $M^\pi = X^*$. Hence, if $(X \xrightarrow{F} Y)$ satisfies the PMP, then, by Theorem 3 (ii), $X^* = M^\pi \subset F^*(Y^*)$, so $F^*(Y^*) = X^*$. Consequently, $F(X)$ is closed in Y, (Theorem 4, Page 488 [10]).

(*iii*) If F is compact then so is F^*, and hence $\overline{F^*(Y^*)}$ is separable. But, by our assumption, X^* is non-separable. Therefore $\overline{F^*(Y^*)} \neq X^*$, whence, by part (*i*) above, $(X \xrightarrow{F} Y)$ does not satisfy the PMP, which completes the proof.

We shall give now using Theorem 3(*i*), an example that the converse of Theorem 2 is not valid.

Example 1 [2]: Let $X = Y = c_0$, where c_0 be the normed linear space (c_0) equipped with the norm $\|x\| = \sup_n |\xi_n|$ where $x = \xi_n \in c_0$ [page 35, [10]]. and let

$$F(x) = \sum_{i=1}^{\infty} \left(\frac{1}{2^i}\right) f_i(x)x_i, \quad x \in E \tag{13}$$

where, $\{x_n\}$ denotes the unit vector basis of $X = c_0$ and $\{f_n\}$ denotes the sequence of coordinate functionals on X [so $f_i(x_j) = \delta_{ij}$]. Then F is one-to-one and

$$F^*(g) = \sum_{i=1}^{\infty} (\tfrac{1}{2}i) g(x_i) f_i, \quad g \in Y^* = X^* \tag{14}$$

on the other hand, it is well known (see Remark (*i*) to Theorem 5 [6]) that

$$M^{\pi} = lin\{f_n\},$$

the linear span of $\{f_n\}$. Thus by (14), $X^* = M^{\pi} \subset F^*(Y^*)$, whence, by Theorem 3(*i*), $(X \xrightarrow{F} Y)$ satisfies the PMP. However, by (13), $\overline{F(X)} = X = Y, F(X) \neq X$, so $F(X)$ is not closed, which shows that the converse of Theorem 2 is not valid. Note also that , in this example, F is compact, but X^* is separable and X is not smooth and not semi-reflexive [see Corollary 3 (*ii*) and (*iii*) above].

Now F is one-to-one in Theorem 3, Corollary 3, and Example 1 and the fact that $F(X) = Y$ in Example 1 are not accidental, since the following theorem shows that the general case can be reduced to the case when F has both of these properties.

Theorem 4 *For a linear system* $(X \xrightarrow{F} Y)$*, the following statements are equivalent:*

(i) $(X \xrightarrow{F} Y)$ *satisfies the PMP*

(ii) $(X/KerF \xrightarrow{\tilde{F}} Y)$ *satisfies the PMP where* $\tilde{F}$ *is the (one-to-one) mapping induced by* F*, that is*

$$\tilde{F}(x + KerF) = F(x), x + KerF \in X/KerF; \tag{15}$$

(iii) $(X \xrightarrow{F} \overline{F(X)}$ *satisfies the PMP.*

Proof: By Theorem 1, we have (ii) iff, for each $x_0 + KerF \in X/KerF$, there exists $g_0 \in Y^*$ such that

$$\begin{aligned} &\sup_{\psi\in(Ker\tilde{F})^{\perp}} \left\{|\langle x_0 + KerF, \psi\rangle| : \sup_{x\in X\subset X^{**}} |\langle \psi, x\rangle| = 1\right\} \\ = \ &|\langle x_0 + KerF, \tilde{F}^*(g_0)\rangle| : \sup_{x\in X\subset X^{**}} |\langle \tilde{F}^*(g_0), x\rangle| \end{aligned} \tag{16}$$

But, since $\tilde{F}$ is one-to-one [indeed, $\tilde{F}(x + KerF) = 0$ implies, by (15), $x \in KerF$, whence $x + KerF = KerF$, the zero element of $X/KerF$], we have $(Ker\tilde{F})^{\perp} = (X/KerF)^*$; hence, by the canonical linear isometry (11) of $(KerF)^{\perp}$ onto $(X/KerF)^*$, the left hand sides of (3) and (17) coincide.

Furthermore, by (15), we have

$$\begin{aligned} |\langle x_0 + KerF, \tilde{F}^*(g_0)\rangle| &= |\langle \tilde{F}^*(x_0 + KerF), g_0\rangle| \\ = |\langle F(x_0), g_0\rangle| &= |\langle x_0, F^*(g_0)\rangle| \end{aligned} \tag{17}$$

so $\tilde{F}^*(g_0)$ is the image of $F^*(g_0)$ by the canonical linear isometry (11) of $(KerF)^{\perp}$ onto $(X/KerF)^*$, and therefore

$$\sup_{x\in X\subset X^{**}} |\langle \tilde{F}^*(g_0), x\rangle| = \sup_{x\in X\subset X^{**}} |\langle F^*(g_0), x\rangle|$$

This together with (17), shows that the right-hand side of (3) and (16) coincide, too. Thus $(i) \Leftrightarrow (ii)$.

Finally, the equivalence $(i) \Leftrightarrow (iii)$ follows from the definition (1) of the PMP and the observations that $g_0 \mid_{F(X)} \in (\overline{F(X)})^*$ for each $g_0 \in Y^*$ and that, conversely, each $\gamma_0 \in (\overline{F(X)})^*$ can be extended to a $g_0 \in Y^*$. This completes the proof of Theorem 4.

Combining Theorem 4 and Corollary 3 (i) and (ii), we obtain the following generalization of Corollary 3(i) and (ii).

Corollary 4 *Let* $(X \xrightarrow{F} Y)$ *be a linear system, with* X *smooth. Then*

(i) *if* $(X \xrightarrow{F} Y)$ *satisfies the PMP,* $\overline{F^*(Y^*)} = (KerF)^{\perp}$;

(ii) *if* $(X \xrightarrow{F} Y)$ *satisfies the PMP and* X *is semireflexive,* $F^*(Y^*) = (KerF)^{\perp}$ *and* $F(X)$ *is closed in* Y.

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Flow of a Bingham Fluid by means of Peristaltic Waves

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Abstract: The problem of peristaltic induced flow of a Bingham fluid under long wavelength approximation is investigated. It is found that the pressure drop increases with the flow rate and there exists a linear relation between them. For any given set of other parameters, the magnitude of the pressure drop is higher for Bingham fluid than Newtonian fluid. Pressure drop decreases with increasing amplitude ratio but increases with the yield stress. The friction force possesses the character similar to the pressure drop.

1 Introduction

Fluids transport by means of peristaltic waves has been the subject of engineering and scientific research since the first invstigation by Latham [1]. It is a form of fluid transport that occurs when a progressive wave of area contraction or expansion propagates along the length of a distensible tube containing liquid or mixture. Besides its applications to physiological flows [2], the study of the mechanism of peristaltic pumping is of significant use to engineering works. To understand peristaltic action in different situations, several theoretical and experimental studies have been conducted in the literature. An account of the most of the investigations reported so far may be had from Srivastava and Coworkers [2,3].

Excepting a few, most of the studies in the literature have been carried out for Newtonian fluids. It is known that a large number of fluids of practical importance including the physiological fluids behave like non-Newtonian fluids. The studies [2–7] have considered this aspect of the problem since the initial investigations by Raju and Devanathan [8,9]. The research reported here considers the peristaltic transport of a Bingham fluid, a special case of a non-Newtonian fluid which inhibits yield stress.

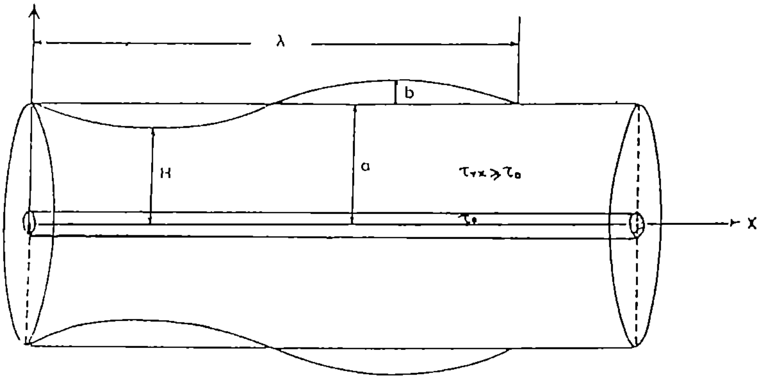

Fig 1 Geometrical representation of peristaltic waves in a circular tube .

2 Formulation and Analysis

Consider the axisymmetric flow in a circular tube with a sinusoidal wave travelling down its wall. The geometry of the wall surface is described as (Fig. 1)

$$H(x,t) = a + b\sin\frac{2\pi}{\lambda}(x-ct), \tag{1}$$

where a is the radius of the tube, b is the amplitude of the peristaltic wave,

λ is the wavelength, c is the wave propagation velocity, t is the time and x is the axial coordinate.

Assuming that the circular tube is filled with a homogeneous and incompressible Bingham fluid, the appropriate equation in the wave frame of reference (moving with speed c) under long wavelength approximation and neglecting the inertia terms [10], is expressed as

$$\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rx}) = -\frac{dp}{dx} \tag{2}$$

where $r\tau_{rr}$, for a Bingham fluid, is given by

$$-\frac{\partial u}{\partial r} = \frac{\tau_{rr} - \tau_0}{\mu}; \quad \tau_{rx} \geq \tau_0, \tag{3}$$

$$\frac{\partial u}{\partial r} = 0; \quad \tau_{rr} \leq \tau_0, \tag{4}$$

where p is the pressure, μ is the velocity of the fluid, τ_0 is the yield stress, r is the radial coordinate and u is the axial velocity of the fluid. The relation (4) corresponds to the vanishing of the velocity gradient in the region in which $\tau_{rx} \leq \tau_0$ and implies a plug flow.

An introduction of the following dimensionless variables

$$r' = \frac{r}{a},\ x' = \frac{x}{\lambda},\ t' = \frac{ct}{\lambda},$$
$$u' = \frac{u}{c},\ \tau_0{}' = \frac{a\tau_0}{c\mu},\ p' = \frac{a^2}{c\mu\lambda}p, \tag{5}$$

into equations (2)-(4), after dropping primes, yields

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-\frac{\partial u}{\partial r} + \tau_0\right)\right] = -\frac{dp}{dx}; \quad \tau_{rr} \geq \tau_0, \tag{6}$$

$$\frac{\partial u}{\partial r} = 0; \quad \tau_{rx} \leq \tau_0, \tag{7}$$

where τ_{rx} and τ_0 are dimensionless shearing and yield stresses respectively.

The non-dimensional boundary conditions are

$$u = -1 \text{ at } r = H/a = h = 1 + \phi \sin 2\pi x, \tag{8}$$

$$\tau_r\text{, is finite at } r = 0 \text{ to } a_p, \tag{9}$$

where $\phi = b/a < 1$ and a_p is the radius of plug flow region.

The straight forward integration of equation (6) under the boundary conditions (8) and (9), yields the expression for velocity, u as

$$u = -1 + \frac{1}{(dp/dx)}\left[\left(\frac{r}{2}\frac{dp}{dx} + \tau_0\right)^2 - \left(\frac{h}{2}\frac{dp}{dx} + \tau_0\right)^2\right]. \tag{10}$$

The constant plug flow velocity $u = u_p$ is obtained from equation (10) evaluated at $r = a_p$, a_p being the non-dimensional radius of plug flow region and is given by [11]

$$\tau_0 = -\frac{dp}{dx}\frac{a_p}{2} \tag{11}$$

which may be conveniently calculated using the expression for the flow rate.

Considering the radius of plug flow area to be small as compared to the non-plug flow area [3,13,14], one obtains the expression for the instantaneous volumetric flow rate Q $(= Q'/\pi a^2 c$, Q' being the flux in the moving system) as

$$\begin{aligned} Q &= 2\int_0^h r u\,dr \\ &= -h^2 + \frac{h^3\tau_h}{4}\left\{\left(1 - \frac{\tau_0}{\tau_h}\right)^2\left[1 + \frac{2}{3}\frac{\tau_0}{\tau_h} + \frac{1}{3}\left(\frac{\tau_0}{\tau_h}\right)^2 \right.\right. \\ &\quad \left.\left. - \frac{1}{3}\left(\frac{\tau_0}{\tau_h}\right)^4\right]\right\} \end{aligned} \tag{12}$$

where $\tau_h = (-h/2)(dp/dx)$ is the shear stress at the tube wall. Under the condition [13,14], $\tau_0/\tau_h \ll 1$, equation (12) yields

$$-\frac{dp}{dx} = 8\,\lfloor_{u'}(Q + h^2) + \frac{1}{3}\frac{\tau_0}{h}\Big], \tag{13}$$

Now following the report by Shapiro et al. [10], one derives the mean volume flow rate, $\overline{Q}$ over a period as

$$\overline{Q} = Q + 1 + \frac{\phi^2}{2}. \tag{14}$$

An use of equations (13) and (14) into equation (11) determines the radius of plug flow region in its non-dimensional form as

$$a_p = \frac{3\tau_0 h^4}{4[3(\overline{Q} - 1 - \phi^2 + h^2) + \tau_0 h^3]}, \tag{15}$$

which reveals that the plug flow radius depends on yield stress, the tube geometry and the flow rate. a fact which has been previously observed in [2,12,15] etc.

Since the pressure drop, $\Delta p = p(0) - p(\lambda)$, across one wave length is same whether measured in moving or stationary coordinate system, it is therefore calculated using equation (14) into (13) as

$$\begin{aligned} \Delta p &= -\int_0^1 \frac{dp}{dx} dx \\ &= \frac{8}{(1-\phi^2)^{7/2}} \left[\frac{1}{3}(1-\phi^2)^3 \tau_0 + \overline{Q}(1 + \frac{3}{2}\phi^2) \right. \\ &\quad \left. + \frac{\phi^2}{4}(\phi^2 - 16) \right], \end{aligned} \tag{16}$$

The dimensionless friction force per wavelength, F ($= F'/\pi\lambda c\mu$, F' being the friction force and is same whether measured in moving or stationary system) at the wall is given as

$$\begin{aligned} F &= \int_0^1 h^2 \left(-\frac{dp}{dx}\right) dx \\ &= \frac{8}{(1-\phi^2)^{3/2}} \left[\overline{Q} - 1 - \frac{\phi^2}{2} + \left(1 + \frac{\tau_0}{3}\right)(1-\phi^2)^{3/2} \right]. \end{aligned} \tag{17}$$

The pressure drop for zero time-mean flow, $(\Delta p)_{\overline{Q}=0}$ and the time-mean flow for zero pressure drop, $(\overline{Q})_{\Delta p=0}$ which are of particular interest, from equation (16) are obtained as

$$(\Delta p)_{\overline{Q}=0} = \frac{8}{(1-\phi^2)^{7/2}}\left[\frac{1}{3}(1-\phi^2)^3\tau_0 + \frac{\phi^2}{4}(\phi^2-16)\right], \qquad (18)$$

$$(\overline{Q})_{\Delta p=0} = \frac{1}{(2+3\phi^2)}\left[\phi^2\left(8-\frac{\phi^2}{2}\right) + \frac{2}{3}(\phi^2-1)^3\tau_0\right], \qquad (19)$$

In absence of the plug flow (i.e., $\tau_0 = 0$), equations (16)–(19) yield the corresponding results for a Newtonian fluid as

$$\Delta p = \frac{8}{(1-\phi^2)^{7/2}}\left[\overline{Q}(1+\frac{3}{2}\phi^2) + \frac{\phi^2}{4}(\phi^2-16)\right], \qquad (20)$$

$$F = \frac{8}{(1-\phi^2)^{3/2}}\left[\overline{Q} - 1 - \frac{\phi^2}{2} + (1-\phi^2)^{3/2}\right], \qquad (21)$$

$$(\Delta p)_{\overline{Q}=0} = \frac{2\phi^2(\phi^2-16)}{(1-\phi^2)^{7/2}}, \qquad (22)$$

$$(\overline{Q})_{\Delta p=0} = \frac{\phi^2(16-\phi^2)}{2(2+3\phi^2)}. \qquad (23)$$

Equations (20)–(23) are the same results as derived from the analysis of Shapiro et al. [10], Jaffrin and Shapiro [16] and Barton and Raynor [17].

3 Numerical Results and Discussion

In order to discuss the results of the analysis quantitatively, computer codes were developed for the numerical evaluations of the analytical results obtained for dimensionless pressure drop and the friction force (equations (14) and (15)). Some of the important results are displayed graphically in Figs. 2–6.

The average pressure drop, Δp versus the time-mean flow rate, $\overline{Q}$ is plotted for different values of the parameters ϕ and τ_0, which shows a linear relation between them. As expected, an increase in the flow rate increases the pressure drop and thus minimum flow rate is achieved at zero

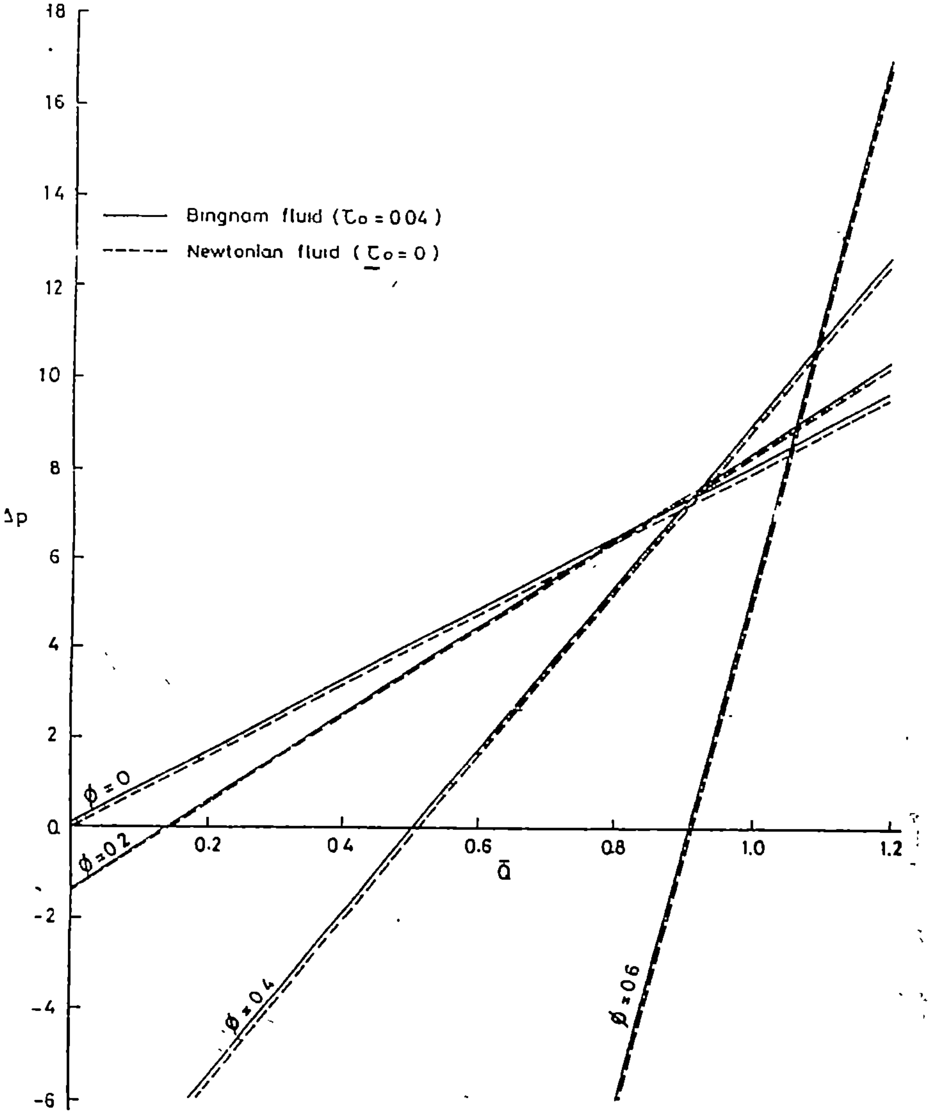

Figure 2. Pressure - flow rate relationship for different φ and τo.

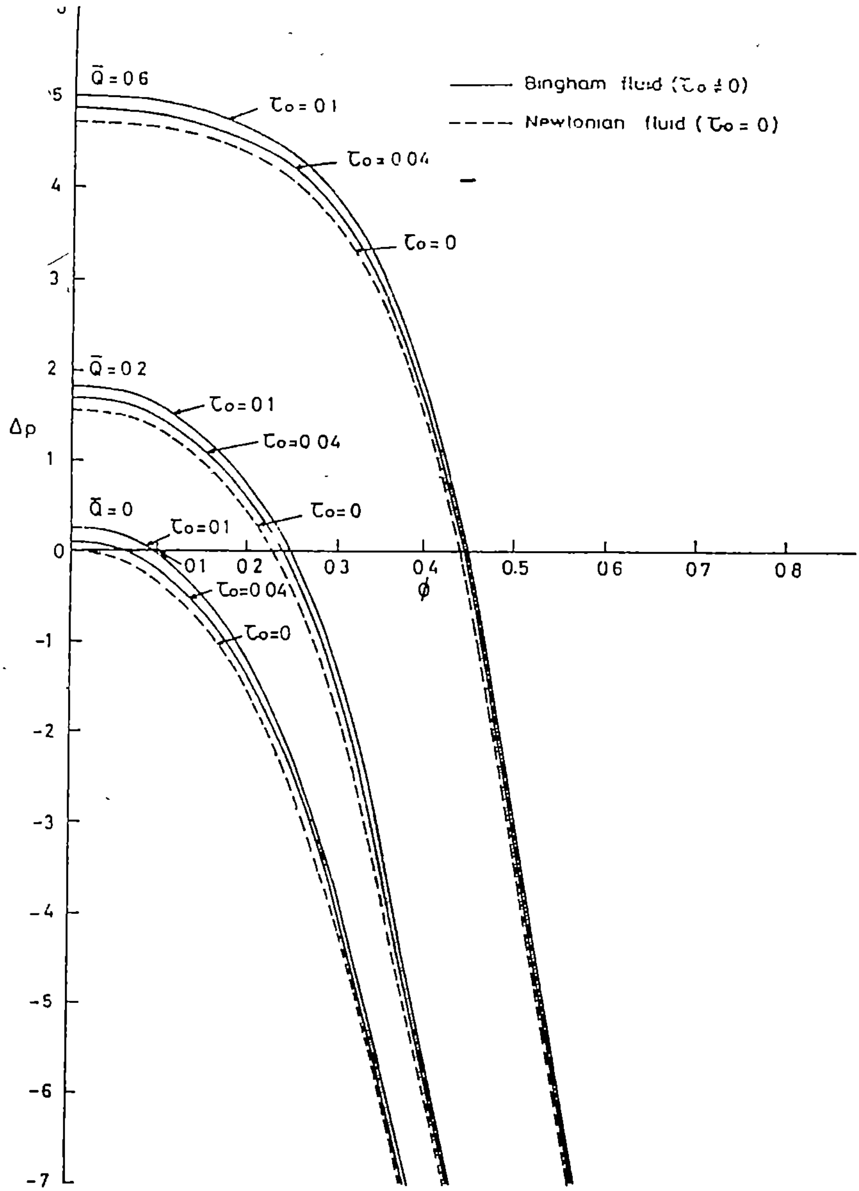

Figure 3 Variation of pressure drop , Δ p with ϕ for different Q and τ_o

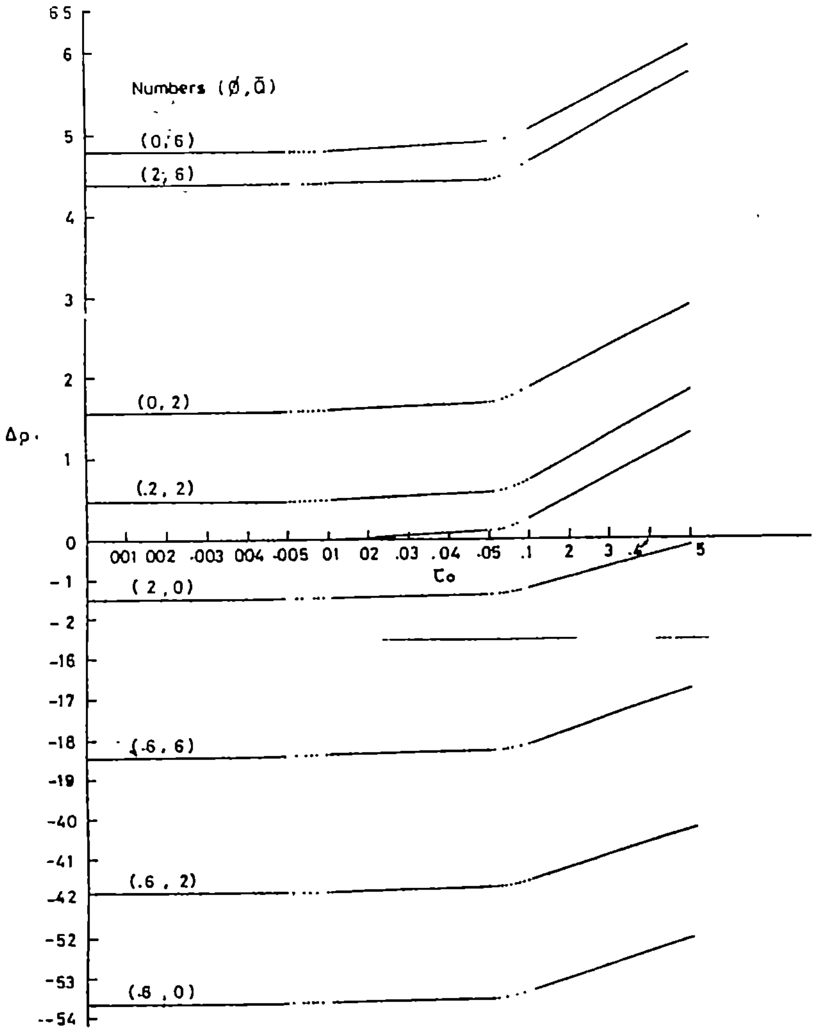

Figure [illegible] Variation of pressure drop, Δp with yield stress, τ_o for different ϕ and $\bar{Q}$

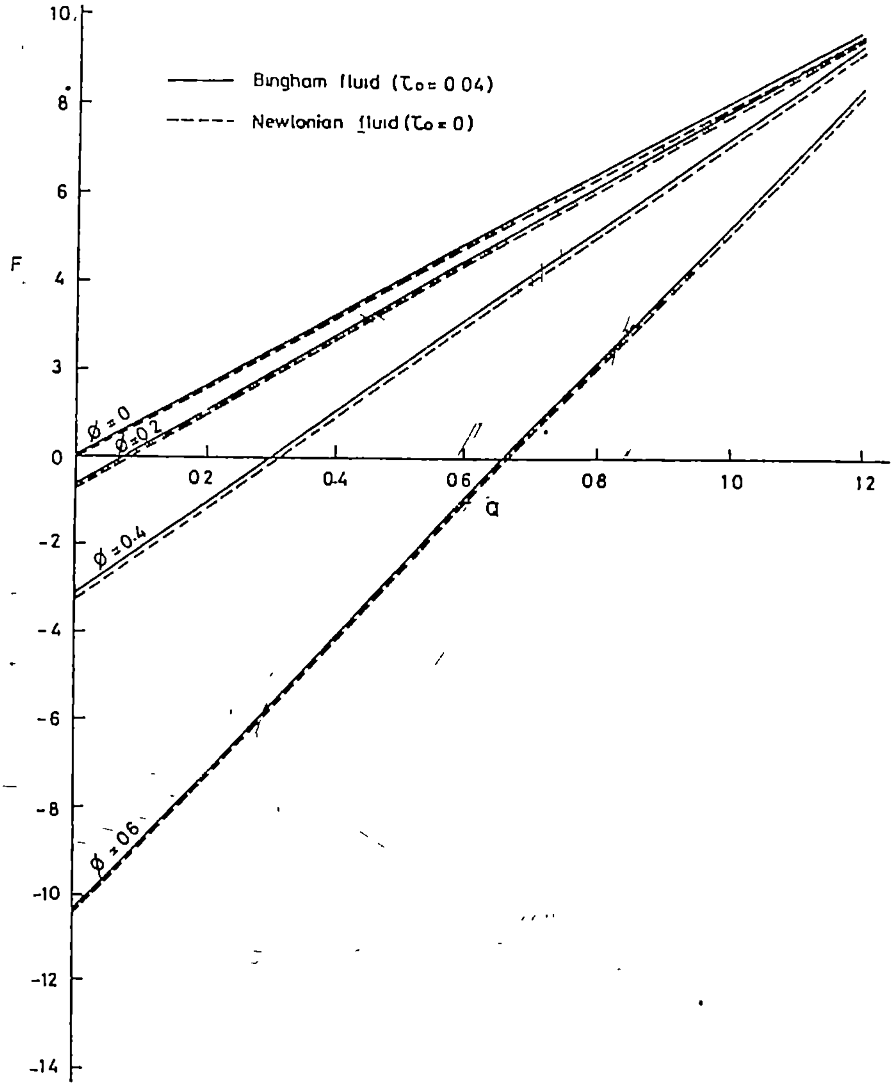

Figure 5. Friction force-flow rate relationship for different ϕ and τ_o.

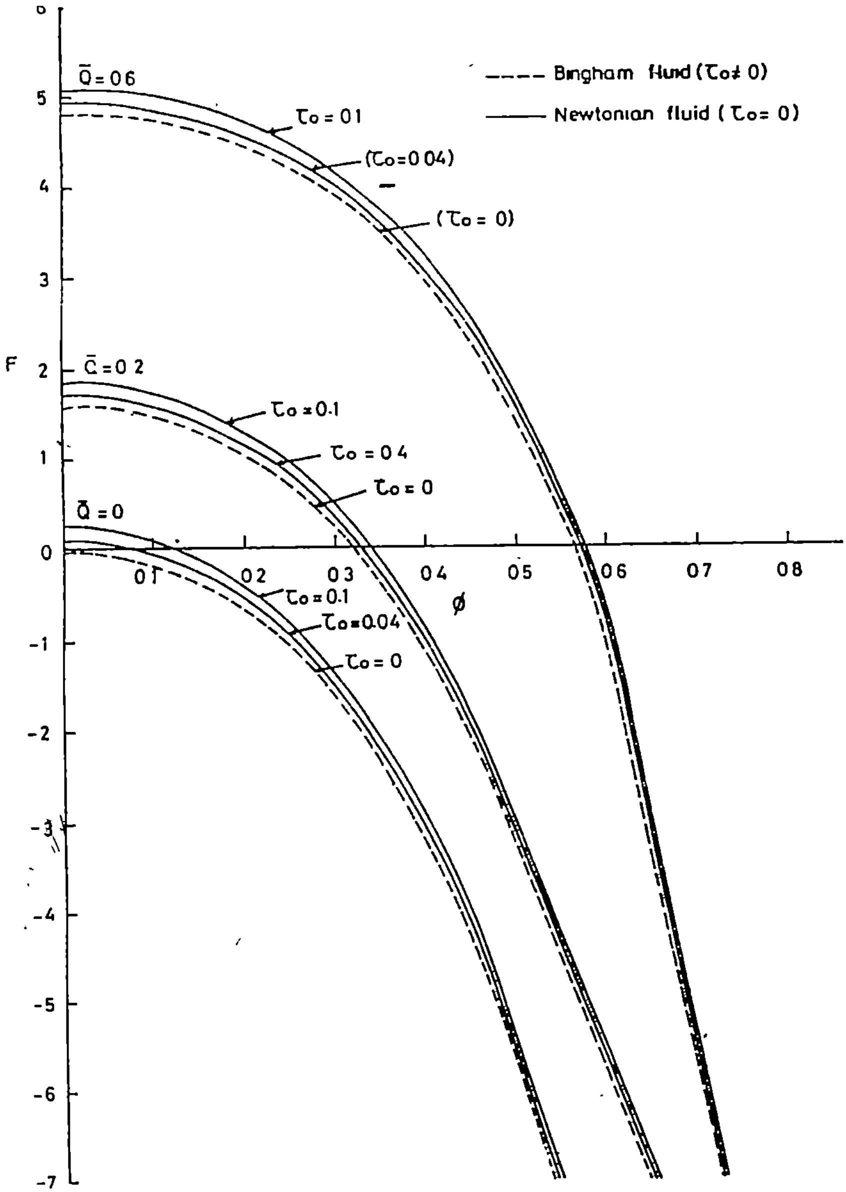

Figure 6. Variation of friction force, F with ϕ for different $\bar{Q}$ and τ_o.

pressure drop and the minimum pressure drop occurs at zero flow rate. The magnitude of the pressure drop under a given set of parameters is greater for a Bingham fluid than for a Newtonian fluid. The variation of pressure drop Δp with amplitude ratio ϕ is shown in Fig. 3. The pressure drop decreases indefinitely with increasing amplitude ratio for any given set of other parameters. One notices from Fig. 4 that pressure drop, Δp increases with the yield stress, τ_0 and for any given set of other parameters, its minimum value is achieved at zero yield stress (Newtonian fluid). A study of Figs. 5 and 6 reveals that the friction force, F possesses a character similar to the pressure drop, Δp (an opposite character to the pressure rise $-\Delta p$).

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